King Saud University
Department of Mathematics
M-203
(Differential and Integral Calculus)
Second-Mid Term Examination
(Second Semester 1433/1434)

Max. Marks: 25 Time: 90 minutes

Marking Scheme: All questions carry equal marks.

Q. No: 1 Evaluate the integral $\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{1+y^5} dy dx$.

Q. No: 2 Find the **surface area** of the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$.

Q. No: 3 Find the **mass** of a **lamina** that has the shape of the region bounded by graphs of the equations $y = e^x$, $x = 0$, $x = 1$, $y = 0$ and has the **area mass density** $\delta(x, y) = xy$.

Q. No: 4 Sketch the region bounded by the graphs of the equations $z = 25 - x^2$, $z = 0$, $y = 0$, $y = 3$ and use a triple integral to find its volume.

Q. No: 5 Use **spherical coordinates** to evaluate the integral

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) dz\, dx\, dy .$$

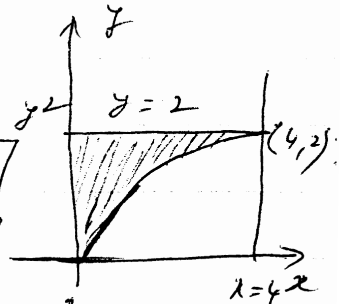
M-203

II Mid-term Exam. (II Sem. 1433/1434)

Time: 90 Minutes Max. Marks: 25

Q#1) Evaluate the integral $\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{1+y^5}\, dy\, dx$ [Marks: 5]

Soln: we reverse the order of integration and we have



(2) $\int_0^2 \int_0^{y^2} \frac{x}{1+y^5}\, dx\, dy = \int_0^2 \frac{1}{1+y^5}\left[\frac{x^2}{2}\right]_0^{y^2}$

$= \frac{1}{2}\int_0^2 \frac{y^4}{1+y^5}\, dy = \frac{1}{10}\ln|1+y^5|\Big]_0^2$ (2)

$= \frac{1}{10}\ln|33|$ (1)

$y = \sqrt{x}$

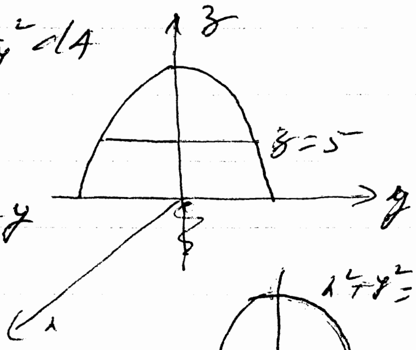
$\therefore x = y^2$

$0 \le y \le 2$

$0 \le x \le y^2$

Q#2) Find the surface area of the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$. [Marks: 5]

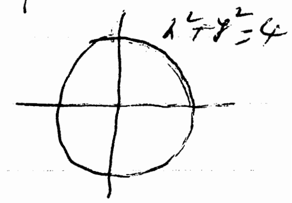
Soln. Surface Area $S = \iint_{R_{xy}} \sqrt{1 + g_x^2 + g_y^2}\, dA$



we have $z = 9 - x^2 - y^2 = g(x,y)$

(1) $g_x = -2x$ and $g_y = -2y$

$= \iint_{R_{xy}} \sqrt{1 + 4x^2 + 4y^2}\, dA$ (1)



(2) $= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\, r\, dr\, d\theta$

$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3}(1+4r^2)^{3/2}\Big]_0^2 = \frac{1}{12}(17^{3/2} - 1)\, 2\pi$

$\simeq 36.2$ (1)

Q#3) Find the mass of a lamina that has the shape of the region bounded by the graphs of the equations $y = e^x$, $x = 0$, $x = 1$, $y = 0$ and has the area mass density $\delta(x, y) = xy$. [Marks: 5]

Soln: Mass of the lamina: $m = \int_0^1 \int_0^{e^x} xy \, dy\, dx$ (2)

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^{e^x} dx$$

$$= \int_0^1 x \left[\frac{e^{2x}}{2}\right] dx \quad \text{Integrate by parts.}$$

$$= \frac{1}{2} x e^{2x} \cdot \frac{1}{2} \Big]_0^1 - \frac{1}{2} \int_0^1 \frac{e^{2x}}{2} \, dx$$

$$= \frac{1}{4} e^2 - \frac{1}{4} \int_0^1 e^{2x} \, dx$$ (2)

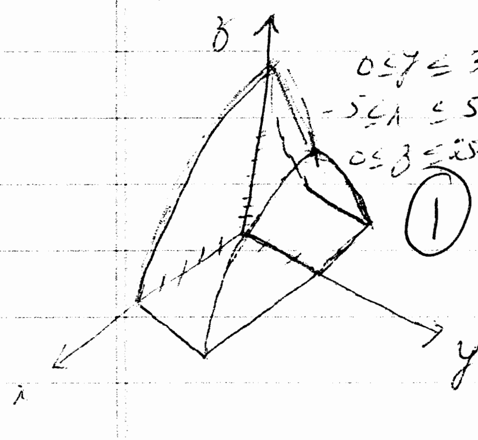
$$= \frac{1}{4} e^2 - \frac{1}{8} \left[e^{2x}\right]_0^1$$

$$= \frac{1}{4} e^2 - \frac{1}{8}\left[e^2 - 1\right]$$

$$= \frac{1}{8} e^2 - \frac{1}{8} = \frac{1}{8}(e^2 - 1)$$ (1)

Q#4) Sketch the region bounded by the graphs of the equations $z = 25 - x^2$, $z = 0$, $y = 0$, $y = 3$ and use a triple integral to find its volume. [Marks: 5]

Soln. Volume $V = \int_0^3 \int_{-5}^5 \int_0^{25-x^2} dz\, dx\, dy$ (3)



$0 \le y \le 3$
$-5 \le x \le 5$
$0 \le z \le 25 - x^2$

(1)

$$= \int_0^3 \int_{-5}^5 (25 - x^2)\, dx\, dy = \int_0^3 \left[25x - \frac{x^3}{3}\right]_{-5}^5 dy$$

$$= \int_0^3 \left(125 + 125 - \frac{125}{3} - \frac{125}{3}\right) dy$$

$$= \left(250 - \frac{250}{3}\right)(3) = 500$$ (1)

③ ○

Q #5) Use spherical Coordinates to evaluate the integral

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dx\, dy$$

[Marks:5]

Soln. we have $0 \le \rho \le \sqrt{18}$

$0 \le \varphi \le \pi/4$

$0 \le \theta \le \pi/2$

$$= \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{18}} \rho^4 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

③

$$= \int_0^{\pi/2} \int_0^{\pi/4} \frac{\rho^5}{5}\Big]_0^{\sqrt{18}} \sin\varphi\, d\varphi\, d\theta$$

$$\frac{\sqrt{18}(324)}{5} \int_0^{\pi/2} \int_0^{\pi/4} \sin\varphi\, d\varphi\, d\theta = \frac{324\sqrt{18}}{5} \int_0^{\pi/2} -\cos\varphi\Big]_0^{\pi/4} d\theta$$

$$= \frac{324\sqrt{18}}{5} \int_0^{\pi/2} \left(1 - \frac{1}{\sqrt{2}}\right) d\theta$$ ①

$$= \frac{\overset{162}{\cancel{324}}\sqrt{18}}{5} \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\pi}{\cancel{2}}$$

$$= \frac{162\sqrt{18}}{5} \left(1 - \frac{1}{\sqrt{2}}\right)\pi$$ ①