M-203, FIRST MIDTERM EXAMINATION
(Semester-II, 1438-1439)
Department of Mathematics, College of Science
KING SAUD UNIVERSITY

Max Marks-25 Time: 90 Min.

Q.1 Test the convergence or divergence of the sequence

$$\left\{\frac{7^{-n}}{\csc n}\right\}$$

and if it is converges, find its limit. [4]

Q.2 Find the sum of the series [5]

$$\sum_{n=1}^{\infty}\left[\left(\frac{e}{\pi}\right)^{n}+\frac{2}{(n+1)(n+3)}\right].$$

Q.3 Determine whether the following series

$$\sum_{n=2}^{\infty}\frac{(-1)^{n}}{\ln n}$$

is conditionally convergent, absolutely convergent or divergent. [5]

Q.4 Find the interval of convergence for the power series [5]

$$\sum_{n=2}^{\infty}\frac{(-1)^{n}}{(2n+1)}x^{2n+1}.$$

Q.5 Find the Maclaurin series of the function $f(x) = \cos 2x$ and use first three nonzero terms to approximate the value of the following integral to four decimal places [6]

$$\int_{0}^{0.1}\sin^{2} x dx.$$

M-203

I Mid-term Exam. (II Semester 1438/1439)

Max. Marks: 25 Time 90 Minute

Q#1. Test the Convergence or divergence of the sequence $\left\{ \frac{7^{-n}}{\csc n} \right\}$ and, if it converges, find its limit.

Soln. We have $a_n = \frac{7^{-n}}{\csc n} = \frac{\sin n}{7^n}$

we know $-1 \le \sin n \le 1$

$$\Rightarrow \lim_{n\to\infty} \frac{-1}{7^n} \le \lim_{n\to\infty} \frac{\sin n}{7^n} \le \lim_{n\to\infty} \frac{1}{7^n}$$

Hence, by Sandwich Theorem. $\lim_{n\to\infty} \frac{\sin n}{7^n} = 0$; Conv.

Q#2) Find the Sum of the Series $\sum_{n=1}^{\infty} \left[\left(\frac{e}{\pi}\right)^n + \frac{2}{(n+1)(n+3)} \right]$

Soln. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n + \sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$

$\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ which is a Conv. Geom. Series

$$\text{Sum} = \frac{\frac{e}{\pi}}{1-\frac{e}{\pi}} = \frac{e}{\cancel{\pi}} \cdot \left(\frac{\cancel{\pi}}{\pi - e}\right) = \frac{e}{\pi - e}$$

$$\frac{2}{(n+1)(n+3)} = 2\left[\frac{A}{n+1} + \frac{B}{n+3}\right]$$

$$\frac{\cancel{2}}{\cancel{(n+1)(n+3)}} = \cancel{2}\left[\frac{A(n+3)+B(n+1)}{\cancel{(n+1)(n+3)}}\right]$$

Or, $1 = A(n+3) + B(n+1)$

Put $n=-1$. we have $1 = A(2) \therefore A = \frac{1}{2}$

Put $n=-3$, we have $1 = -2B \therefore B = -\frac{1}{2}$

$$\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$$

$$S_n = \sum_{n=1}^{N} \frac{2}{(n+1)(n+3)} = \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right) + \left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$$

$$= \left(\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{n+3}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)} = \lim_{n\to\infty}\left[\frac{5}{6}-\frac{1}{n+3}\right]$$

$$= \frac{5}{6}$$

Hence Sum $S = \frac{e}{\pi - e} + \frac{5}{6}$

Q#3) Discuss whether the Series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally Convergent or absolutely Conv or divergent

Soln. $\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\frac{1}{\ln n} \geq \frac{1}{n}$ $\forall n$ where $\sum \frac{1}{n}$ is divergent

Hence by BCT, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

But $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ by AST, we have

(i) $\frac{1}{\ln n}$ is clearly decreasing and

(ii) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, hence Conv.

Therefore, $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ is conditionally Conv.

Q#4) Find the interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$$

Soln. We apply the absolute Ratio test.

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}\, x^{2n+3}}{2n+3} \times \frac{2n+1}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \overbrace{\left(\frac{2n+1}{2n+3}\right)}^{=1} |x^2| = |x^2| \Rightarrow |x| < 1 \text{ for abs. conv.}$$

$$\Leftrightarrow -1 < x < 1$$

At $x=-1$, we have $\sum_{n=2}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{(2n+1)} = \sum_{n=2}^{\infty} \frac{(-1)^n (-1)}{2n+1}$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{2n+1} \text{ which is conv by A.S.T.}$$

At $x=1$, we have $\sum_{n=2}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1}$ which is also conv. by A.S.T.

Hence interval of conv: $[-1, 1]$.

Q#5) Find the Maclaurin Series of the function $f(x) = \cos 2x$ and use it to approximate the value of the integral taking three non-zero terms to four decimal places.

$$\int_0^{0.1} \sin^2 x \, dx$$

Soln. We have $f(x) = \cos 2x$ and we know

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{2n!} + \cdots$$

$$= \sum (-1)^n \frac{x^{2n}}{}$$

Hence $f(x) = \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$

$$+ (-1)^n \frac{(2x)^{2n}}{2n!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{2n!}$$

Hence, $\sin^2 x = \frac{1 - \cos 2x}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2x)^{2n}}{2\cdot(2n!)}$

$$= x^2 - \frac{x^4}{3} + \frac{64 x^6}{720} - \cdots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots$$

$$\therefore \int_0^{0.1} \sin^2 x \, dx \approx \int_0^{0.1} \left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} \cdots\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5(3)} + \frac{2x^7}{7\times 45} \cdots\right]_0^{0.1}$$

$$= \frac{0.001}{3} - \frac{0.00001}{15} + 2\times\frac{0.0000001}{315}$$

$$= 0.0003$$

0.0003

3.334×10^{-4}