King Saud University
Department of Mathematics
M-203
(Differential and Integral Calculus)
Second Mid-Term Examination
(Summer Semester 1433/1434)

Max. Marks: 25 Time: 90 Minutes

Marking Scheme: **Q.1**(5), **Q.2**:(5), **Q.3:**(5), **Q.4**:(5), **Q.5**:(5)

Q. No: 1 Evaluate the integral

$$\int_0^1 \int_{y^{1/2}}^1 \sqrt{y}\, e^{x^2}\, dx\, dy.$$

Q. No: 2 Find the **area of the surface** $z = 1 - x - y$ lying above the circle $x^2 + y^2 = 2x$.

Q. No: 3 Find the mass of the cube with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ if its density at a point (x, y, z) is proportional to the square of its distance from the origin.

Q. No: 4 Evaluate the integral $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$ by changing to cylindrical coordinates.

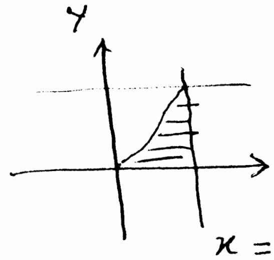
Q. No: 5 If spherical coordinates of a point are $(\rho, \varphi, \theta) = \left(2, \frac{\pi}{3}, \frac{\pi}{4}\right)$, find its

(a) rectangular coordinates, and
(b) cylindrical coordinates.

Q.No 1 $\displaystyle\int_0^1\int_{y^{3/2}}^1 \sqrt{y}\, e^{x^2}\,dx\,dy$ [Marks: 5]

$0 \le y \le 1$

$y^{3/2} \le x \le 1$



$$= \int_0^1\int_0^{x^{2/3}} \sqrt{y}\, e^{x^2}\,dy\,dx \quad (3) \quad = \int_0^1 \left[\frac{y^{3/2}}{3/2}\right]_0^{x^{2/3}} e^{x^2}\,dx$$

$$= \frac{2}{3}\int_0^1 (x^{2/3})^{3/2} e^{x^2}\,dx = \frac{2}{3}\int_0^1 x\, e^{x^2}\,dx$$

$0 \le x \le 1$

$0 \le y \le x^{2/3}$

$$= \frac{1}{3}\int_0^1 e^{x^2}(2x)\,dx = \frac{1}{3}\left[e^{x^2}\right]_0^1 = \frac{1}{3}(e-1) \quad (2)$$

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Q.2 [Marks: 5]

$$S.A. = \iint_{R_{xy}} \sqrt{1+g_x^2+g_y^2}\,dA \quad (1)$$

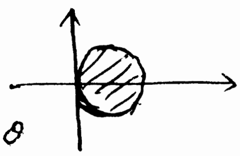
$z = 1-x-y = g(x,y) \Rightarrow g_x = 1,\ g_y = 1$

Region $x^2+y^2 = 2x$

$r = 2\cos\theta$

$0 \le r \le 2\cos\theta$

$-\pi/2 \le \theta \le \pi/2$



$$= \iint_{R_{xy}} \sqrt{3}\,dA \quad (1)$$

$$= \sqrt{3}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta \quad (2)$$

$$= \sqrt{3}\int_{-\pi/2}^{\pi/2} \left.\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \frac{\sqrt{3}}{2}\int_{-\pi/2}^{\pi/2} [2\cos\theta]^2\,d\theta = \frac{\sqrt{3}}{2}\int_{-\pi/2}^{\pi/2} 4\cos^2\theta\,d\theta$$

$$= 2\sqrt{3}\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \sqrt{3}\left[\theta + \sin 2\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \sqrt{3}\left[\pi/2 + \pi/2\right] = \sqrt{3}\,\pi \quad (1)$$

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Q: 3 $\text{mass} = \iiint_Q \delta \, dv$ [Marks: 5] $\delta \propto (x^2+y^2+z^2)$

$\delta = k(x^2+y^2+z^2)$ 1

$$= k\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dx\,dy = k\int_0^1\int_0^1 \left[x^2z + y^2z + \frac{z^3}{3}\right]_0^1 dx\,dy$$

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$$= k\int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right)dx\,dy = k\int_0^1 \left[\frac{x^3}{3} + y^2x + \frac{x}{3}\right]_0^1 dy$$

$$= k\int_0^1 \left[\frac{1}{3} + y^2 + \frac{1}{3}\right]dy = k\left[\frac{2y}{3} + \frac{y^3}{3}\right]_0^1 = k\left[\frac{2}{3}+\frac{1}{3}\right] = k$$ 1

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Q: 4 $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$$

[Marks: 5]

$-2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$\sqrt{x^2+y^2} \le z \le 2$

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2 r\,dz\,dr\,d\theta$$ 3

$\Rightarrow \begin{cases} r \le z \le 2 \\ 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$

$$= \int_0^{2\pi}\int_0^2 r^3 [z]_r^2\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (2r^3 - r^4)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{2} - \frac{r^5}{5}\right]_0^2 d\theta = \int_0^{2\pi}\left[\frac{16}{2} - \frac{32}{5}\right]d\theta = \left(8 - \frac{32}{5}\right)2\pi = \left(\frac{8}{5}\right)2\pi = \frac{16\pi}{5}$$

2

Q: 5 [Marks: 5]

$(\rho, \varphi, \theta) = (2, \pi/3, \pi/4) \Rightarrow \rho = 2,\ \varphi = \pi/3,\ \theta = \pi/4$

$z = \rho\cos\varphi = 2\cos(\pi/3) = 2(1/2) = 1$

$r = \rho\sin\varphi = 2\sin(\pi/3) = 2(\sqrt{3}/2) = \sqrt{3}$

$\Rightarrow (r, \theta, z) = (\sqrt{3}, \pi/4, 1)$ 2

$x = r\cos\theta = \sqrt{3}\cos(\pi/4) = \sqrt{3}/\sqrt{2}$

$y = r\sin\theta = \sqrt{3}\sin(\pi/4) = \sqrt{3}/\sqrt{2}$

$\Rightarrow (x, y, z) = (\sqrt{3}/\sqrt{2}, \sqrt{3}/\sqrt{2}, 1)$

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