M-203, SECOND MIDTERM EXAMINATION
(Semester-II, 1438-1439)
Department of Mathematics, College of Science
KING SAUD UNIVERSITY

Max Marks-25 Time: 90 Min

Note: *All questions carry equal marks*

Q.1 Evaluate the integral

$$\int_0^1 \int_x^{\sqrt[3]{x}} e^{\frac{1}{4}y^2(2-y^2)}\,dy\,dx.$$

Q.2 Find the surface area of the portion of the surface $z = 2x^2 + 2y^2$ that lies inside the cylinder $x^2 + y^2 = 1$.

Q.3 Find the mass of the circular lamina $x^2 + y^2 = 1$ having density

$$\delta(x, y) = x^2.$$

Q.4 Find the center of mass of the solid bounded by the graphs of the equations $z = \sqrt{x^2 + y^2}$ and $z = 2 - \sqrt{x^2 + y^2}$, if the density at point $P(x, y, z)$ of the solid is directly proportional to the distance of P from the z-axis.

Q.5 Evaluate the integral

$$\iiint_{\mathbf{Q}} \sqrt{x^2 + y^2 + z^2}\,dV,$$

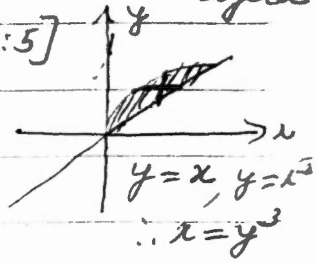
where $\mathbf{Q}$ is the region that is inside the cone $z = \sqrt{x^2 + y^2}$ and the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

M- 203

II Mid-term Exam. (II semester 1438/1439)

Time: 90 Mts. Max. Marks: 25

Q#1) Evaluate the Integral $\int_0^1 \int_x^{\sqrt[3]{x}} e^{\frac{1}{4}y^2(2-y^2)}\, dy\, dx$ [Marks: 5]



Soln. Given $x \le y \le x^{\frac{1}{3}}$, $0 \le x \le 1$ } vertical strip.

we change it to:

$y^3 \le x \le y$, $0 \le y \le 1$ } Horizontal strip.

Hence, we have $\int_0^1 \int_{y^3}^{y} e^{\frac{1}{4}y^2(2-y^2)}\, dx\, dy$

1+2=3

$= \int_0^1 (y-y^3)\, e^{\frac{1}{4}y^2(2-y^2)}\, dy$

Put $t = \frac{1}{4}y^2(2-y^2)$

$dt = (y-y^3)\, dy$

$= \int_0^{1/4} e^t\, dt$

$= \left[e^t\right]_0^{1/4} = \cancel{\left[e^{\frac{1}{2}y^2 - \frac{1}{4}y^4}\right]_0^1}$

$= \cancel{e^{\frac{1}{2}-\frac{1}{4}}} - 1 = \left(e^{\frac{1}{4}} - 1\right)$ ②

Q#2) Find the Surface area of the portion of the Surface $z = 2x^2 + 2y^2$ that lies inside the cylinder $x^2+y^2=1$ [5 Marks]

Soln. we have $z = 2x^2+2y^2 = f(x,y)$

$\Rightarrow f_x = 4x$ and $f_y = 4y$ ①

$\therefore$ S.A. $= \iint_R \sqrt{1+16x^2+16y^2}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{1+16r^2}\, r\, dr\, d\theta$ ③

Put $1+16r^2 = t$ ∋: $32r\, dr = dt$

$= \int_0^{2\pi}\int_1^{17} \sqrt{t}\cdot\frac{1}{32}\, dt = \int_0^{2\pi} \frac{1}{32}\left[\frac{2}{3} t^{3/2}\right]_1^{17} d\theta$

$= \frac{1}{48}\left[(1+16r^2)^{3/2}\right]_0^1 = \frac{1}{48}(17^{3/2}-1)\int_0^{2\pi} d\theta = \frac{1}{48}(17^{3/2}-1)\cdot 2\pi$

$= (17^{3/2}-1)\frac{\pi}{24}$ // ①

Q#3) Find the mass of the circular lamina $x^2+y^2 \le 1$ having density $\delta(x,y) = x^2$ [Marks: 5]

Soln. Mass: $m = \int_0^{2\pi}\int_0^1 r^2\cos^2\theta \, r\,dr\,d\theta$ (3)

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 \cos^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2\times 4}2\pi + \frac{1}{4}\cdot\frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{\pi}{4}$$ (2)

Q#4) Find the centre of the mass of the solid bounded by the graphs of the equations $z=\sqrt{x^2+y^2}$ and $z = 2-\sqrt{x^2+y^2}$, if the density at point $P(x,y,z)$ of the solid is directly proportional to the distance of P from the z-axis. [Marks: 5]

Soln. $\delta(x,y,z) = k\sqrt{x^2+y^2} = kr$ (1)

$$m = k\int_0^{2\pi}\int_0^1\int_r^{2-r} r\cdot r\,dz\,dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^1 (2-r-r)r^2\,dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^1 2(1-r)r^2\,dr\,d\theta = 2k\int_0^{2\pi}\left(\frac{1}{3}-\frac{1}{4}\right)d\theta$$

$$= 2k\cdot 2\pi\left(\frac{1}{12}\right) = \frac{k\pi}{3}$$ (2)

$$M_{xy} = k\int_0^{2\pi}\int_0^1\int_r^{2-r} z\,r\,r\,dz\,dr\,d\theta = \frac{1}{2}k\int_0^{2\pi}\int_0^1\left[(2-r)^2-r^2\right]r^2\,dr\,d\theta$$

$$= \frac{1}{2}k\int_0^{2\pi}\int_0^1(4-4r+r^2-r^2)r^2\,dr\,d\theta$$

$$= \frac{1}{2}\times 4k\int_0^{2\pi}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1 = 2k\cdot 2\pi\times\frac{1}{12} = \frac{k\pi}{3}$$

$\therefore \bar{z} = \frac{M_{xy}}{m} = \frac{k\pi}{3}\times\frac{3}{k\pi} = 1$

By Sym: $\bar{x}=\bar{y}=0$ $\therefore (\bar{x},\bar{y},\bar{z}) = (0,0,1)$ (2)

Q#5) Evaluate the Integral $\iiint_Q \sqrt{x^2+y^2+z^2}\, dV$,

where Q is the region that is inside the cone $z=\sqrt{x^2+y^2}$ and the hemisphere $z=\sqrt{1-x^2-y^2}$. [Marks: 5]

Soln. $\iiint_Q \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$ (4)

$= \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{\rho^4}{4}\right]_0^1 \sin\varphi\, d\varphi\, d\theta$

$= \frac{1}{4}\int_0^{2\pi} \left[-\cos\varphi\right]_0^{\pi/4} d\theta = \frac{1}{4_2} \times 2\pi \left[-\frac{1}{\sqrt{2}}+1\right]$

$= \frac{\pi}{2}\left(\frac{-1+\sqrt{2}}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\pi(\sqrt{2}-1)$ (1)