King Saud University
Department of Mathematics
M-203 (First Mid-Term)
(Differential and Integral Calculus)
(Summer semester 1435/1436)

Max. Marks: 25 Time: 90 minutes

Marking Scheme: All Questions carry equal marks

Q. No: 1 Determine whether the sequence $\left\{\sqrt{n^2+3n}-n\right\}_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

Q. No: 2 Determine whether the following series is absolutely convergent, conditionally convergent, or divergent $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{1}{n(\ln n)^2}$.

Q. No: 3 Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{n^2+5n+6}+\frac{2}{5^{n-1}}\right]$.

Q. No: 4 Find the **interval of convergence** and **radius of convergence** of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{n(x+3)^n}{5^n}$.

Q. No: 5 Find the Maclaurin series for $f(x)=e^x$ and approximate the integral $\displaystyle\int_0^1 e^{-x^2}dx$ using three non-zero terms.

Q#1) Determine whether the sequence $\{\sqrt{n^2+3n}-n\}_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

Soln.
$$\lim_{n\to\infty}\frac{\{(\sqrt{n^2+3n}-n)(\sqrt{n^2+3n}+n)\}}{(\sqrt{n^2+3n}+n)} \quad 2$$

$$=\lim_{n\to\infty}\frac{\cancel{n^2}+3n-\cancel{n^2}}{\sqrt{n^2+3n}+n}=\lim_{n\to\infty}\frac{3\cancel{n}}{\cancel{n}\{\sqrt{1+\frac{3}{n}}+1\}} \quad 2$$

$$=\frac{3}{2};\ \text{Conv.} \quad 1$$

Q#2) Determine whether the following series is absolutely convergent, conditionally convergent or divergent

$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{n(\ln n)^2}$$

Soln.
$$\lim_{n\to\infty}\left|(-1)^n\frac{1}{n(\ln n)^2}\right|=\lim_{n\to\infty}\frac{1}{n(\ln n)^2}$$

$$f(x)=\frac{1}{x(\ln x)^2} \quad 1$$

$f(x)$ is real valued, continuous and decreasing in $[2,\infty)$. We apply Integral test. 1

$$\lim_{t\to\infty}\int_2^t\frac{1}{x(\ln x)^2}\,dx$$

Put $\ln x=u \Rightarrow \frac{1}{x}dx=du$

$$\int\frac{du}{u^2}=-\frac{1}{u}=-\frac{1}{\ln x}$$

$$=\lim_{t\to\infty}\left[-\frac{1}{\ln x}\right]_2^t$$

$$=\lim_{t\to\infty}\left[-\frac{1}{\ln t}+\frac{1}{\ln 2}\right]=\frac{1}{\ln 2};\ \text{Conv.} \quad 1$$

2

Q#3) Find the Sum of the Series $\sum_{n=1}^{\infty}\left[\frac{1}{n^2+5n+6}+\frac{2}{5^{n-1}}\right]$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2+5n+6} + 2\sum_{n=1}^{\infty}\frac{1}{5^{n-1}}$$

$\sum_{n=1}^{\infty}$ $\frac{1}{n^2+5n+6} = \frac{1}{(n+2)(n+3)} = \frac{A}{n+2}+\frac{B}{n+3}$

n^2+5n+6

$= n^2+3n+2n+6$

$= n(n+3)+2(n+3)$

$= (n+2)(n+3)$

$$= \frac{A(n+3)+B(n+2)}{(n+2)(n+3)}$$

Put $n=-2$. we have $1=A$

Put $n=-3$. we have $1=-B \therefore B=-1$

$$\therefore \frac{1}{n^2+5n+6} = \frac{1}{n+2}-\frac{1}{n+3}$$

$$\sum_{n=1}^{n}\frac{1}{n^2+5n+6} = \sum_{1}^{n}\left[\frac{1}{n+2}-\frac{1}{n+3}\right]$$

$$=\left[\left(\frac{1}{3}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\frac{1}{6}\right)+\cdots + \left(\frac{1}{n+2}-\frac{1}{n+3}\right)\right]$$

$$=\left(\frac{1}{3}-\frac{1}{n+3}\right) \quad 2$$

$\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{n+3}\right) = \frac{1}{3}$, Conv.

$= S_1$ (Say) 1

Also, $\sum_{n=1}^{\infty}\frac{2}{5^{n-1}} = 2\sum_{n=1}^{\infty}\frac{1}{5^{n-1}}$ This is a Geom. Series with $|r|=\frac{1}{5}<1 \Rightarrow$ Conv.

$$\therefore S_2 = 2\left[\frac{1}{1-\frac{1}{5}}\right] = 2\left[\frac{5}{4}\right] = \frac{5}{2} \quad 1$$

$$\therefore S = S_1+S_2 = \frac{1}{3}+\frac{5}{2} = \frac{2+15}{6} = \frac{17}{6} \approx 2.83$$

1

(3)

Q#4) Find the Interval of convergence and radius of conv of the power series $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

Soln. $\lim_{n\to\infty} \left| \frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \times \frac{5^n}{n(x+3)^n} \right|$

$= \frac{1}{5}|x+3|$

For abs. conv. $\frac{1}{5}|x+3| < 1 \iff |x+3| < 5$

$\iff -5 < x+3 < 5$

$\iff -8 < x < 2$ (2)

At $x = -8$, we have $\sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n}$, Div. by ~~nth term test~~ AST. (1)

At $x = 2$, we have $\sum_{n=0}^{\infty} \frac{n\,5^n}{5^n}$; Div. by nth term ~~test~~ (1)

Interval of conv: $(-8, 2)$; Radius of conv: $\frac{2-(-8)}{2} = 5$ (1)

Q#5) Find the Maclaurin Series for $f(x) = e^x$ and approximate the Integral $\int_0^1 e^{-x^2}\,dx$ using three non-zero terms.

Soln. we have $f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$

Hence $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$ (2)

$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots$ (1)

$\therefore \int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} + \dots\right)dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \dots\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{5(2)}$ (1)

$= 1 - \frac{1}{3} + \frac{1}{10} = \frac{30-10+3}{30} = \frac{23}{30} \approx 0.7666\ldots$ (1)