King Saud University
Department of Mathematics
M-203
(Differential & Integral Calculus)
First Mid-Term Examination
(II-Semester 1437/38)

Max. Marks: 25

Time: 90 minutes

Marks: Q.1[4]; Q.2[5]; Q.3[4]; Q.4[6]; Q.5[6]

Q.No: 1 Discuss the convergence or divergence of the sequence $\left\{\left(\frac{n+4}{n+2}\right)^{n^2}\right\}_{n=1}^{\infty}$.

Q. No: 2 Find the **sum** of the series $\sum_{n=1}^{\infty}\left[\frac{2}{n(n+2)}+\left(\frac{e}{\pi}\right)^n\right]$.

Q. N0: 3 Determine whether the following series is **absolutely convergent, conditionally convergent, or divergent** $\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{\sqrt{n^2+n}}$.

Q.No: 4 Find the interval of convergence and the radius of convergence of the power series: $\sum_{n=2}^{\infty}(-8)^n\frac{x^{3n}}{\ln n}$.

Q. N0: 5 Find the Maclaurin series for the function $f(x) = \sin x$ and use its first three non-zero terms to approximate the integral:

$$\int_{-1}^{1}\frac{\sin x}{x}dx .$$

Q:1 $\lim_{n\to\infty} \left(\frac{n+4}{n+2}\right)^{n^2}$ 1^∞ - form [Marks 4]

$$\ln y = n^2 \ln\left(\frac{1+4/n}{1+2/n}\right) \quad 1$$

$$= \frac{\ln(1+4/n) - \ln(1+1/n)}{1/n^2} \quad \frac{0}{0} -$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{1}{1+4/x}(-4/x^2) - \frac{1}{1+1/x}(-1/x^2)}{-2/x^3} \quad 2$$

$$= \lim_{x\to\infty} \frac{\frac{-4}{1+4/x} + \frac{1}{1+1/x}}{-2/x} = -\frac{3}{0}$$

$$= \infty$$

$$\Rightarrow \lim y = e^\infty = \infty \quad \text{dgt} \quad 1$$

Q:2 $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} + \sum_{n=1}^{\infty} (e/\pi)^n$ [Marks. 5]

$$= \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right] + \frac{e/\pi}{1-e/\pi} \quad 1$$

$$s_n = \left[\frac{1}{1} - \cancel{\frac{1}{2}}\right] + \left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right] + \dots + \left[\cancel{\frac{1}{n}} - \frac{1}{n+1}\right]$$

$$= 1 - \frac{1}{n+1} \longrightarrow 1 \quad 3$$

$$s = 1 + \frac{e}{\pi - e} \quad 1$$

Q: 3 $\sum_{n=1}^{\infty} \left| \frac{(-\frac{1}{2})^n}{\sqrt{n^2+n}} \right| = \sum_{n=1}^{\infty} \frac{1}{2^n(\sqrt{n^2+n})}$ [Marks 4]

Comparing with $\sum_{n=1}^{\infty} \frac{1}{2^n}$ 2

$\frac{1}{2^n\sqrt{n^2+n}} < \frac{1}{2^n}$ for all $n \geq 1$

2

$\Rightarrow$ c'gt $\Rightarrow$ A.C.

Q: 4 $\sum_{n=2}^{\infty} (-8)^n \frac{x^{3n}}{\ln n} \Rightarrow \left| \frac{u_{n+1}}{u_n} \right| = 8 \frac{\ln n}{\ln(n+1)} |x^3|$

[Marks 6]

$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = 8|x^3| \Rightarrow$ c'gt if $8|x^3| < 1$

$\Rightarrow |x^3| < 1/8 \Rightarrow |x| < \frac{1}{2} \Rightarrow \boxed{-\frac{1}{2} < x < \frac{1}{2}}$ 3

At $x = -\frac{1}{2}$ $\sum_{n=2}^{\infty} \frac{(-8)^n(-\frac{1}{8})^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ Comparing $\sum_{n=2}^{\infty} \frac{1}{n}$

1 $\frac{1}{n} \leq \frac{1}{\ln n}$ for all $n \geq 2$ d'gt

At $x = \frac{1}{2}$ $\sum_{n=2}^{\infty} \frac{(-8)^n(\frac{1}{8})^n}{\ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln 2}$ c'gt by AST

1

$\Rightarrow \boxed{-\frac{1}{2} < x \leq \frac{1}{2}}$ $\boxed{\rho = \frac{1}{2}}$ 1

Q:5 $f(x) = \sin x \Rightarrow f(0) = 0, \; f'(0) = 1, \; f''(0) = 0$

$f'''(0) = -1, \; - - -$ [Marks 6]

$$f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + - - - -$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - - + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + - - -$$ 2

$$\int_{-1}^{1} \frac{\sin x}{x} \, dx \approx \int_{-1}^{1} \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \right] dx$$ 2

$$= \int_{-1}^{1} \left[1 - \frac{x^2}{6} + \frac{x^4}{120}\right] dx$$

$$= \left[x - \frac{x^3}{18} + \frac{x^5}{600}\right]_{-1}^{1}$$

$$= \left[1 - \tfrac{1}{18} + \tfrac{1}{600}\right] - \left[-1 + \tfrac{1}{18} - \tfrac{1}{600}\right]$$

$$= 2 - \tfrac{1}{9} + \tfrac{1}{300} = 2 - 0.1111 + 0.0033$$

$$\approx 1.8922$$ 2