KING SAUD UNIVERSITY **COLLEGE OF SCIENCE**

M203 **DEPARTMENT OF MATHEMATICS** **TIME: 90 Minutes**

(SEMESTER 1, 1439-1440)

First Mid-term Exam

Max Marks:25

Note: All questions carry equal Marks.

Q1. Determine the convergence or divergence of the sequence $\left\{\frac{n}{\sqrt{n}-1} - \frac{n}{\sqrt{n}+1}\right\}$ and if it converges, find the limit.

Q2. Find the sum of the series:

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}.$$

Q3. Determine whether the series: $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$ is absolutely convergent, conditionally convergent or divergent.

Q4. Find the interval of convergence and the radius of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{\ln(n+1)}.$$

Q5. Find the MacLaurin series for e^x and use it to find the MacLaurin series for 10^x and also use its first three non-zero terms to approximate the integral $\int_0^1 10^x\, dx$.

M-203

I Mid-term Exam. (I Sem. 1439/1440)

Solutions to the Questions. Max. Marks: 2

Q#1. Determine the convergence or divergence of the sequence $\left\{\frac{n}{\sqrt{n}-1} - \frac{n}{\sqrt{n}+1}\right\}$ and if it converges, find the limit. [Marks: 5]

Soln. $\frac{n}{\sqrt{n}-1} - \frac{n}{\sqrt{n}+1} = \frac{n(\sqrt{n}+1) - n(\sqrt{n}-1)}{(\sqrt{n}-1)(\sqrt{n}+1)} = \frac{n\sqrt{n}+n-n\sqrt{n}+n}{n-1}$ (3)

$= \lim_{n\to\infty} \frac{2n}{n-1} = 2$; cong. (1) (1)

Q#2) Find the Sum of the Series $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ [Marks: 5]

Soln. $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$ (2)

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$

$= \left(\frac{1}{1^2} - \cancel{\frac{1}{2^2}}\right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$= 1 - \frac{1}{(n+1)^2}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 - \frac{1}{(n+1)^2}\right] = 1 \quad \therefore \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} =$ (2) (1

Q#3) Determine whether the Series $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ is absolutely cong, Cond.-Cong. or divergent. [Marks: 5]

Soln. $\sum_{n=1}^{\infty} \left|(-1)^n \frac{\sqrt{n}}{n+1}\right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} = \sum_{n=1}^{\infty} a_n$ (say)

Choose $\sum_{n=1}^{\infty} b_n = \sum \frac{\sqrt{n}}{n} = \sum \frac{1}{\sqrt{n}}$ which is div. by p-series test

By L.C.T. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+1} \times \sqrt{n}\right) = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$

Hence L.C.T. $\sum_{n=1}^{\infty} \left|(-1)^n \frac{\sqrt{n}}{n+1}\right|$ is divergent (3)

Now, by AST, $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ and $\frac{\sqrt{n}}{n+1}$ is decreasing

Hence, by AST, it is Cong. $\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ is conditionally convergent. 2

Q#4. Find the interval of convergence and the radius of convergence of the power series: $\sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{\ln(n+1)}$ [Marks: 5]

Soln. we apply absolute Ratio test:

$$\lim_{n\to\infty} \left| \frac{(-x)^{n+2}}{\ln(n+2)} \cdot \frac{\ln(n+1)}{(-x)^{n+1}} \right| = |x|$$

For abs. cong. $|x| < 1 \Leftrightarrow -1 < x < 1$ 2

At $x = -1$, we have $\sum_{n=1}^{\infty} \frac{(-(-1))^{n+1}}{\ln(n+1)} = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

which is divergent. (By BCT, $\frac{1}{1+n} \leq \frac{1}{\ln(1+n)}$ $\forall n$

and $\sum_{n=1}^{\infty} \frac{1}{1+n}$ is div. 1

For $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$ which is cong. by AST. 1

Hence the required interval of convergence: $(-1, 1]$

and Radius of convergent: $\frac{1-(-1)}{2} = \underline{\underline{1}}$ 1

Q#5) Find the Maclaurin Series for e^x and use it to find the Maclaurin Series for 10^x and also use its first three non-zero terms to approximate the integral $\int_0^1 10^x\,dx$. [Marks: 5]

Soln.

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x \Rightarrow f'(x) = 1$$

$$f''(x) = e^x \Rightarrow f''(x) = 1$$

$$\vdots$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$$

$$\vdots$$

Hence
$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now,
$$10^x = e^{\ln 10^x} = e^{x\ln 10} = \sum_{n=0}^{\infty} \frac{(x \ln 10)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{(\ln 10)^n x^n}{n!} \quad \text{for all } x.$$

Finally,
$$\int_0^1 10^x\,dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(\ln 10)^n x^n}{n!} \simeq \sum_{n=0}^{2} \int_0^1 \frac{(\ln 10)^n x^n}{n!}$$
$$= \int_0^1 1\,dx + \int_0^1 (\ln 10)x\,dx + \int_0^1 \frac{(\ln 10)^2}{2} x^2\,dx$$
$$= \left[x + \ln 10 \frac{x^2}{2} + \frac{(\ln 10)^2}{6} x^3\right]_0^1$$
$$= 1 + \frac{\ln 10}{2} + \frac{(\ln 10)^2}{6} \approx 3.03494222$$