

King Saud University
Department of Mathematics
M-203
(Differential & Integral Calculus)
First Mid-Term Examination
(I-Semester 1437/38)

Max. Marks: 25

Time: 90 minutes

Marks: Q.1(4); Q.2(4); Q.3(6); Q.4(6); Q.5(5)

Q.No: 1 Determine whether or not the sequence $\left\{ \sqrt{n^2 + n} - n \right\}_{n=1}^{\infty}$ converges, and if it converges, find its limit.

Q. No: 2 Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n + 4^n}{5^n + 7^n}$

Q. No: 3 Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is absolutely convergent, conditionally convergent or divergent.

Q.No: 4 Find the interval of convergence and the radius of convergence of the power

series: $\sum_{n=1}^{\infty} (-2)^n \frac{(x+3)^n}{n}$.

Q. N0: 5 Find the first four non-zero terms of the Taylor series for the function $f(x) = \ln(x)$ in powers of $(x-2)$.

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Q #1) Determine whether or not the sequence $\{\sqrt{n^2+n} - n\}_{n=1}^{\infty}$ converges, and if it converges, find its limit. [Marks: 4]

Sol. $\lim_{n \rightarrow \infty} (\sqrt{n^2+n} - n)$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+n} - n}{\sqrt{n^2+n} + n} = \lim_{n \rightarrow \infty} \frac{n}{n(\sqrt{1+\frac{1}{n}} + 1)} = \frac{1}{2}; \text{ Cong. } \textcircled{1}$$

Q #2) Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n + 4^n}{5^n + 7^n}$. [Marks: 4]

Sol. we apply Limit Comparison test (LCT)
Choose $b_n = \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$ which is a Cong. Geom. series.

$$\lim_{n \rightarrow \infty} \frac{3^n + 4^n}{5^n + 7^n} \times \left(\frac{7}{4}\right)^n = 1 \neq 0 \quad \textcircled{2}$$

$$= \lim_{n \rightarrow \infty} \frac{4^n \left[\left(\frac{3}{4}\right)^n + 1\right]}{7^n \left[\left(\frac{5}{7}\right)^n + 1\right]} \times \left(\frac{7}{4}\right)^n = 1 \neq 0 \quad \textcircled{1}$$

Hence by LCT, $\sum_{n=1}^{\infty} \frac{3^n + 4^n}{5^n + 7^n}$ is also Cong. $\textcircled{1}$

Q#3) Determine whether the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is absolutely convergent, conditionally convergent or divergent. [Marks: 6]

Soln. $\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{n(\ln n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ which is divergent series by Integral test (3)

But $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is cong. by AST (2)

Hence the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)}$ is conditionally cong. (1)

Q#4) Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} (-2)^n (x+3)^n$.

Soln. we apply absolute Ratio test [Marks: 6]

$$\lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1} (x+3)^{n+1}}{(n+1)} \times \frac{n}{(-2)^n (x+3)^n} \right| = 2|x+3|.$$

For absolute cong $|x+3| < 1 \Leftrightarrow -1 < x+3 < 1$
 $\Leftrightarrow -4 < x < -2$ (2)

At $x = -4$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n (-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{2^n}{n}$ (1)

which is a divg. series by n th term test

At $x = -2$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n (1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n}$ (1)

which is divg. by AST, Hence the interval of cong: $(-4, -2)$

Radius: $r = \frac{-2 - (-4)}{2} = 1$ (1)

Please make corresponding change

Q #5) Find the first four non-zero terms of the Taylor Series for the function $f(x) = \ln(x)$ in powers of $(x-2)$. [Marks: 5] ③

Solu. We know the Taylor Series as

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \dots \quad (1)$$

~~Solu.~~ $f(x) = \ln(x) \Rightarrow f(2) = \ln 2$

$$f'(x) = \frac{1}{x} \Rightarrow f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(2) = \frac{2}{8} = \frac{1}{4} \quad (2)$$

Hence, the first four non-zero terms of the Taylor Series

$$f(x) = \ln(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{2!(4)}(x-2)^2 + \frac{1}{3!(4)}(x-2)^3 - \dots$$

$$\therefore f(x) = \ln x = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \dots \quad (2)$$