King Saud University
Department Of Mathematics
M-203
(Differential and Integral Calculus)
First Mid-Term Examination
(I-Semester 1436/1437)

Max. Marks: 25 | Time: 90 Minutes

Marks: Q.No:1[3]; Q.No:2[5]; Q.No:3[5]; Q.No:4[7]; Q.No:5[5]

Q. No: 1 Determine whether or not the sequence $\left\{\ln(2n^2+1)-\ln(n^2+1)\right\}_{n=1}^{\infty}$ converges, and if it converges, find its limit.

Q. No: 2 Test the convergence or divergence of the series

$$\sum_{n=1}^{\infty}\left[\frac{3^{n+2}}{5^n}+\frac{1}{n(n+1)}\right]$$

and find its sum if it converges.

Q: No: 3 Determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent or divergent.

Q. No: 4 Find the interval of convergence and radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{x^n}{n2^n}$

Q, No: 5 Using the first three non-zero terms of the power series, approximate the integral

$$\int_0^{0\ 1} e^{-x^2}dx .$$

M- 203

I Mid-term Exam. (I Semester 1436/1437)

Q#1) Determine whether or not the sequence $\{\ln(2n^2+1)-\ln(n^2+1)\}_{n=1}^{\infty}$ converges, and if it converges, find its limit. [Marks: 3]

Sol^n. we have: $\lim_{n\to\infty} \ln(2n^2+1)-\ln(n^2+1)$

$$=\lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right) \quad (1)$$

$$= \ln 2 \; ; \text{ conv.}$$
(1) (1)

Q#2) Test the convergence or divergence of the series $\sum_{n=1}^{\infty}\left[\frac{3^{n+2}}{5^n}+\frac{1}{n(n+1)}\right]$ [Marks: 5]

and find its sum if it converges.

Sol^n. $\sum_{n=1}^{\infty}\left[\frac{3^{n+2}}{5^n}+\frac{1}{n(n+1)}\right]$

$$=\sum_{n=1}^{\infty}\frac{3^{n+2}}{5^n}+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

Now, $\sum_{n=1}^{\infty}\frac{3^{n+2}}{5^n}=3^2\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$ which is a conv. Geom. series.

Its Sum: $S_1 = 9\left[\frac{\frac{3}{5}}{1-\frac{3}{5}}\right]=9\left[\frac{3}{\cancel{5}}\times\frac{\cancel{5}}{2}\right]$

$$=\frac{27}{2} \quad (2)$$

Also, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$; $a_n=\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$

$\therefore S_1+S_2=\frac{27}{2}+1=\frac{29}{2}=S$ (1)

a_n: $\frac{1}{\cancel{n(n+1)}}=\frac{A(n+1)+Bn}{\cancel{n(n+1)}}$

Put $n=0$. we have $1=A$

Put $n=-1$. we have $1=-B \therefore B=-1$

$\therefore \sum_n a_n = \sum_{k=1}^{n}\left[\frac{1}{n}-\frac{1}{n+1}\right]=\left[\left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\tfrac{1}{4}\right)+\cdots\left(\tfrac{1}{n}-\tfrac{1}{n+1}\right)\right]$

S_2: $\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$ (2)

(2)

Q#3) Determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ is conga. Div.
[Marks: 5]

Soln. $a_n = f(n) = ne^{-n^2} = \frac{n}{e^{n^2}}$

$\therefore f(x) = \frac{x}{e^{x^2}}$.

Clearly: It is positive-value, it is continuous on $[1, \infty)$
we check: It is decreasing:

$f(x) = \frac{x}{e^{x^2}}$

$f'(x) = \frac{e^{x^2} - xe^{x^2}(2x)}{(e^{x^2})^2} < 0 \Rightarrow f\downarrow$ (2)

we apply Integral test:

$$\lim_{t\to\infty} \int_1^t \frac{x}{e^{x^2}}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}\frac{1}{e^{t^2}} + \frac{1}{2}\frac{1}{e}\right]$$

$$= \frac{1}{2e}\text{ ; Conv.}$$ (3)

Put $x^2 = u$.
$2x\,dx = du$
$\Rightarrow x\,dx = \frac{1}{2}du$
$\frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}e^{-u}$

Alternative: Ratio and Root tests are also applicable to this problem #3:

Ratio-test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{n+1}{e^{(n+1)^2}} \times \frac{e^{n^2}}{n}\right] = 0 < 1 \Rightarrow$ Conv.

Root-test $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n \cdot \frac{1}{e^{n^2}}} = 0 < 1 \Rightarrow$ Conv.

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Q#4) Find the Interval of convergence and the radius of conv. of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n\,2^n}$. [Marks: 7]

Soln. we apply Absolute Ratio-test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)2^{n+1}} \times \frac{n \cdot 2^n}{x^n} \right| = \frac{1}{2}|x|$$

For absolute convergence $\frac{1}{2}|x| < 1 \Rightarrow |x| < 2$

$\Leftrightarrow -2 < x < 2$ (4)

At $x = -2$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ which is an convergent Alternating series by AST. (1)

At $x = 2$ we have $\sum_{n=1}^{\infty} \frac{\cancel{2^n}}{n\,\cancel{2^n}}$ which is a div. harmonic series by p-series test. (1)

Interval of conv: $[-2, 2)$

Radius of conv: $r = \frac{2-(-2)}{2} = 2$ (1)

Q#5) Using the first three non-zero terms of the power series, approximate the Integral $\int_0^{0.1} e^{-x^2}\,dx$ [Marks: 5]

Soln. $\int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2!} - \cdots\right) dx$

$= \left[x - \frac{x^3}{3} + \frac{x^5}{5(2)} \cdots \right]_0^{0.1}$ (3)

$= (0.1) - \frac{1}{3}\left(\frac{0.1}{1}\right)^3 + \frac{1}{10}\left(\frac{0.1}{1}\right)^5 \cdots$

$= 0.1 - \frac{1}{3}(0.001) + \frac{1}{10}(0.00001) - \cdots$

$= 0.1 - 0.000333 + 0.000001 - \cdots$

≈ 0.99968 (2)