

King Saud University  
Department of Mathematics  
M-203[Final Examination]  
(Differential and Integral Calculus)  
(Second-Semester 1437/38)

Max.Marks:40

Time:3hrs

**Marking Scheme:** Q.No:1[2+4+3+3]; Q.No:2[4+4+4+4]; Q.No:3[3+3+3+3]

- Q.No: 1 (a) Use integral test to determine whether the series  $\sum_{n=2}^{\infty} \frac{\ln(n^3)}{n}$  converges or diverges.
- (b) Find the interval of convergence and radius of convergence of the power Series  $\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2 3^n}$ .
- (c) Find the power series representation of  $f(x) = \ln(1+x)$ ,  $|x| < 1$  and use first three non-zero terms to calculate  $\ln(1.2)$ .
- (d) Find the Maclaurin series of the function  $f(x) = e^x$  and use its first three non-zero terms to approximate the integral  $\int_0^1 e^{x^2} dx$ .
- Q.No: 2 (a) Evaluate the integral  $\int_0^2 \int_x^2 y^4 \cos(xy^2) dy dx$ .
- (b) Find the area of the surface  $z = 4 - x^2 - y^2$  that lies above the  $xy$ -plane.
- (c) Use cylindrical coordinates to evaluate  $\iiint_Q \sqrt{x^2 + y^2} dV$ , where Q is the solid bounded by the paraboloid  $z = 1 - (x^2 + y^2)$  and the  $xy$ -plane.
- (d) Use spherical coordinates to find the centroid of the solid bounded by the cone  $z = \sqrt{x^2 + y^2}$  and hemi-sphere  $z = \sqrt{8 - x^2 - y^2}$ .
- Q.No: 3 (a) Evaluate the line integral  $\int_C (y^3 + 3y) ds$  where  $C : x = 2 \cos t, y = 2 \sin t, 0 \leq t \leq \pi$ .
- (b) Show that the line integral  $\int_{(0,1,2)}^{(1,0,3)} (2xy + z^2) dx + (x^2 - z + 2y) dy + (2zx - y) dz$  is independent of path and find its value.
- (c) Use the Green's theorem to evaluate  $\oint_C (x^2 y dx + xy^2 dy)$ , where C is the triangle with vertices  $(-1,0)$ ,  $(1,0)$ , and  $(0,1)$ .
- (d) Use the Green's theorem to find the area of the circle  $x^2 + y^2 = a^2, a > 0$ .

①

$$\underline{Q.1(a)} \quad \sum_{n=2}^{\infty} \frac{\ln(n^3)}{n}$$

[Marks: 2]

$$f(x) = \frac{3 \ln x}{x} > 0 \quad \text{on } [2, \infty)$$

(i)  $f(x)$  cont. on  $[2, \infty)$ 

$$(ii) f'(x) = 3 \frac{(1 - \ln x)}{x^2} < 0 \quad \text{on big } x \in [2, \infty)$$

$$3 \int_2^{\infty} \ln x \left(\frac{1}{x}\right) dx = 3 \lim_{t \rightarrow \infty} \int_2^t \ln(x) \left(\frac{1}{x}\right) dx = \frac{3}{2} \lim_{t \rightarrow \infty} \left[ (\ln x)^2 \right]_2^t$$

$$= \infty \quad \text{d'gt} \quad \text{①}$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2 3^n}$$

[Marks: 4]

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(2x-5)^{n+1}}{(n+1)^2 3^{n+1}} \times \frac{n^2 3^n}{(2x-5)^n} \right| = \frac{n^2}{3(n+1)^2} |2x-5|$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{3} |2x-5| \Rightarrow \text{c'gt if } |2x-5| < 3$$

$$\Rightarrow \boxed{1 < x < 4} \quad \text{①}$$

C'gence at  $x=1$ 

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

c'gt AS ①

C'gence at  $x=4$ 

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

c'gt p-series ①

Interval of C'gence  $1 \leq x \leq 4$ Radius of C'gence  $\rho = \frac{3}{2}$  ①

(2)

Q:1 (c)  $\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x [1 - t + t^2 - t^3 + \dots] dt$  [Marks: 3]

$$= \left[ t - \frac{t^2}{2} + \frac{t^3}{3} - \dots \right]_0^x \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

$\ln(1.2) \approx \ln(1+0.2) = (0.2) - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3}$

$$= 0.2 - 0.02 + 0.00267 \approx 0.183$$

Q:1 (d)  $f(x) = e^x, f'(x) = e^x, f''(x) = e^x, \dots$  [Marks: 3]

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

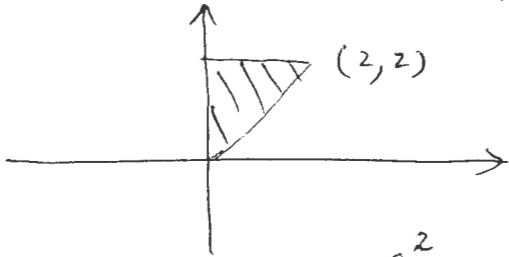
$$e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \dots$$

$$\int_0^1 e^{x^2} dx \approx \int_0^1 [1 + x^2 + \frac{x^4}{2}] dx = \left[ x + \frac{x^3}{3} + \frac{x^5}{10} \right]_0^1$$

$$= 1.4333$$

Q:2 (a)  $\int_0^2 \int_x^2 y^4 \cos(xy^2) dy dx$   $0 \leq x \leq 2$  [Marks: 4]  
 $x \leq y \leq 2$

$0 \leq y \leq 2$   
 $0 \leq x \leq y$



$$= \int_0^2 \int_0^y y^4 \cos(xy^2) dx dy = \int_0^2 y^4 \left[ \frac{\sin(xy^2)}{y^2} \right]_0^y dy$$

$$= \int_0^2 y^2 \sin(y^3) dy \quad u = y^3$$

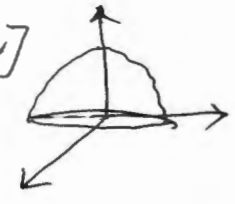
$$= \frac{1}{3} \int_0^8 \sin u du = -\frac{1}{3} [\cos u]_0^8 = \frac{1}{3} [1 - \cos 8]$$

(3)

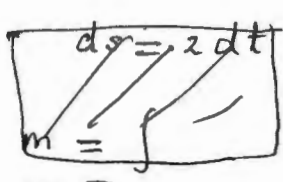
Q: 2(b)

$$\begin{aligned}
 \text{S.A.} &= \int_0^{2\pi} \int_0^2 \sqrt{1+4x^2+4y^2} \, dA = \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2} \, r \, dr \, d\theta \quad [\text{Marks: 4}] \\
 &= \frac{1}{8} \int_0^{2\pi} \left[ \frac{(1+4r^2)^{3/2}}{3/2} \right]_0^2 d\theta = \frac{1}{12} [17^{3/2} - 1] 2\pi \\
 &= \frac{1}{6} (17^{3/2} - 1) \pi \quad (1)
 \end{aligned}$$

Q: 2(c)

$$\begin{aligned}
 &\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} \sqrt{r^2} \, r \, dz \, dr \, d\theta \quad [\text{Marks: 4}] \\
 &= \int_0^{2\pi} \int_0^1 r^2 (1-r^2) \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{r^3}{3} - \frac{r^5}{5} \right]_0^1 d\theta = \frac{4}{15} \pi \quad (1)
 \end{aligned}$$


Q: 2(d)



$$\begin{aligned}
 m &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{8}} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \quad [\text{Marks: 4}] \\
 &= \int_0^{2\pi} \int_0^{\pi/4} \left[ \frac{\rho^3}{3} \right]_0^{\sqrt{8}} \sin \varphi \, d\varphi \, d\theta = \frac{8\sqrt{8}}{3} \int_0^{\pi/4} [-\cos \varphi]_0^{\pi/4} d\theta = \frac{8\sqrt{8}}{3} \left[ 1 - \frac{1}{\sqrt{2}} \right] 2\pi \quad (1)
 \end{aligned}$$

To calculate  $\bar{z}$

$$\begin{aligned}
 &\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{8}} \rho^3 \sin \varphi \cos \varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \left[ \frac{\rho^4}{4} \right]_0^{\sqrt{8}} \sin \varphi \cos \varphi \, d\varphi \, d\theta \\
 &= \frac{64}{4} \int_0^{2\pi} \left[ \frac{\sin^2 \varphi}{2} \right]_0^{\pi/4} d\theta = 8 \left( \frac{1}{2} \right) 2\pi = 8\pi
 \end{aligned}$$

$$\begin{aligned}
 \bar{z} &= \frac{1}{2\sqrt{8} \left[ 1 - \frac{1}{\sqrt{2}} \right]} \quad , \quad \bar{x} = 0, \quad \bar{y} = 0 \\
 &= \frac{4}{3} (\sqrt{2} - 1) \quad (1)
 \end{aligned}$$

(4)

Q: 3(a)

$$ds = z dt$$

[Marks: 3]

$$\int_0^{\pi} (8 \sin^3 t + 8 \sin t) z dt = 16 \int_0^{\pi} (1 - \cos^2 t) \sin t dt + 12 \int_0^{\pi} \sin t dt$$

$$u = \cos t$$

$$= -16 \int_1^{-1} (1 - u^2) du + 12 [-\cos t]_0^{\pi} \quad (2)$$

$$= -16 \left[ u - \frac{u^3}{3} \right]_1^{-1} + 12 [-(-1) + 1] = -16 \left[ \left(-1 + \frac{1}{3}\right) - \left(1 - \frac{1}{3}\right) \right]$$

$$= -16 \left(-2 + \frac{2}{3}\right) + 24 = \frac{64}{3} + 24 = \frac{136}{3} \quad (1)$$

Q: 3(b)

$$M = 2xy + z^2, \quad N = x^2 z + 2y, \quad P = 2zx - y \quad [\text{Marks: 3}]$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = -1 = \frac{\partial P}{\partial y}$$

⇒ Conservative.

$$f_x = 2xy + z^2 \Rightarrow f(x, y, z) = x^2 y + z^2 x + c_1(y, z)$$

$$f_y = x^2 z + 2y \Rightarrow f(x, y, z) = x^2 y - zy + y^2 + c_2(x, z)$$

$$f_z = 2zx - y \Rightarrow f(x, y, z) = z^2 x - yz + c_3(x, y)$$

$$\Rightarrow f(x, y, z) = x^2 y + z^2 x - zy + y^2 + c \quad (2)$$

$$\int_{(0,1,2)}^{(1,9,3)} = 10 \quad (1)$$

(5)

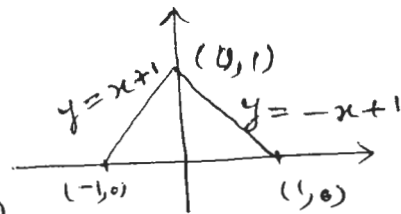
Q. 3(c)

$$\oint_C x^2 y \, dx + xy^2 \, dy$$

[Marks: 3]

$$= \iint_R (y^2 - x^2) \, dA$$

$$= \int_0^1 \int_{y-1}^{1-y} (y^2 - x^2) \, dx \, dy \quad (2)$$



$$= \int_0^1 \left[ y^2 x - \frac{x^3}{3} \right]_{y-1}^{1-y} dy = 2 \int_0^1 \left[ y^2 - y^3 + \frac{(y-1)^3}{3} \right] dy$$

$$= 2 \left[ \frac{y^3}{3} - \frac{y^4}{4} + \frac{(y-1)^4}{12} \right]_0^1 = 2 \left[ \left( \frac{1}{3} - \frac{1}{4} \right) - \frac{(-1)^4}{12} \right]$$

$$= 2 \left( \frac{1}{3} - \frac{1}{4} - \frac{1}{12} \right) = 2 \left( \frac{4-3-1}{12} \right) = 0 \quad (1)$$

Q. 3(d)

$$A = \oint_C x \, dy \quad (1) \quad [\text{Marks: 3}] \quad \begin{aligned} x &= a \cos \theta \\ y &= a \sin \theta \end{aligned}$$

 $2\pi$  $0 \leq \theta \leq 2\pi$ 

$$= \int_0^{2\pi} a \cos \theta \cdot a \cos \theta \, d\theta \quad (1)$$

$$= a^2 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{a^2}{2} \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi}$$

$$= \frac{a^2}{2} (2\pi) = a^2 \pi \quad (1)$$