King Saud University
Department of Mathematics
M-203[Final Examination]
(Differential and Integral Calculus)
(Second-Semester 1434/35)

Max.Marks:40 Time:3hrs

Marking Scheme: Q.No:1[2+3+4+3]; Q.No:2[3+3+3+3]; Q.No:3[4+4+4+4]

Q.No: 1 (a) Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n}{10^2+2^n}$.

(b) Determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

(c) Find the interval of convergence and radius of convergence of the power Series $\sum_{n=0}^{\infty} \frac{e^n}{n+1}(x-1)^n$.

(d) Use the Maclaurin series for the function $f(x) = \sin x$ and use its first three non-zero terms to approximate the integral $\int_0^{0.1} \frac{\sin x^2}{x^2} dx$.

Q.No: 2 (a) Evaluate the integral $\int_0^{\pi/4} \int_y^{\pi/4} \frac{x \sin x}{x^2+y^2} dx dy$.

(b) Find the area of the surface $x^2+y^2+z^2 = 25$ that lies above the plane $z = 4$.

(c) Set up an iterated triple integral that can be used to find the mass of the solid region bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 4$ with density $\delta(x,y,z) = x^2+y^2$.

(d) Let Q be the solid region that lies inside the cone $z = \sqrt{3x^2+3y^2}$ and hemi-sphere $z = \sqrt{4-x^2-y^2}$. Find the volume of Q.

Q.No: 3 (a) Show that the line integral $\int_{(-1,2)}^{(2,3)} (2xy-3)dx + (x^2+4y^3+5)dy$ is independent of path , and find its Value.

(b) (a) Use Green's theorem to evaluate the line integral $\oint_C (e^x+6xy)dx + (8x^2+\sin y^2)dy$ where C is positively oriented boundary of the region bounded by the circle of radii 1 and 3 centered at the origin and lying in the first quadrant.

(c) If S is the surface of the region bounded by the cone $z = \sqrt{x^2+y^2}$ and the planes $z = 0$ and $z = 2$ $\vec{F}(x,y,z) = (xy)\vec{i} + (yz)\vec{j} + (zx)\vec{k}$. Use the Divergence theorem to find $\iint_S \vec{F} \cdot \vec{n} \, dS$.

(d) Use the Stoke's theorem to evaluate $\iint_S (Curl\, \vec{F}) \cdot \vec{n} \, dS$ where S is the hemisphere $z = \sqrt{1-x^2-y^2}$ and the vector field is given by $\vec{F}(x,y,z) = (y)\vec{i} - (x)\vec{j} + (z)\vec{k}$

Q: 1 (a) $\sum_{n=1}^{\infty} \frac{3^n}{10^2 + 2^n}$ [Mark: 2]

we can use limit comp. test with $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

Applying nth term test (1) $\lim_{n\to\infty} \frac{3^n}{10^2 + 2^n}$ $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{3^x \ln 3}{2^x \ln 2} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x \frac{\ln 3}{\ln 2} = \infty$, d'gt (1)

(b) $\sum_{n=1}^{\infty} n e^{-n^2}$ applying integral test [Mark: 3]

$f(x) = \frac{x}{e^{x^2}}$

(i) $f'(x) = e^{-x^2} + x e^{-x^2}(-2x)$

$= \frac{1 - 2x^2}{e^{x^2}} < 0, \ x > 1 \Rightarrow$ DEC (1)

(ii) $\int_1^{\infty} x e^{-x^2} dx = \lim_{t\to\infty} \int_1^t x e^{-x^2} dx = \lim_{t\to\infty} \left[-\frac{1}{2} e^{-x^2}\right]_1^t$

$= \lim_{t\to\infty} \left[-\frac{1}{2} e^{-t^2} + \frac{1}{2} e^{-1}\right] = \frac{1}{2} e^{-1}$ c'gt (1) (1)

(c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{e^{n+1}(x-1)^{n+1}}{n+2} \times \frac{n+1}{e^n (x-1)^n}\right| = \frac{e(n+1)}{(n+2)} |x-1|$ [Mark: 4]

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = e|x-1| \Rightarrow$ c'gt if $e|x-1| < 1$ (1)

$\Rightarrow |x-1| < \frac{1}{e} \Rightarrow -\frac{1}{e} < x - 1 < 1/e \Rightarrow$ $1 - \frac{1}{e} < x < 1 + 1/e$ (1)

Convergence at $x = 1 - 1/e$

$\sum_{n=1}^{\infty} \frac{e^n (1 - 1/e - 1)^n}{n+1} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+1}$

It is c'gt (1)

Convergence at $x = 1 + 1/e$

$\sum_{n=1}^{\infty} \frac{e^n (1 + 1/e - 1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$

It is d'gt

$\Rightarrow$ Interval of c'gence $1 - 1/e \leq x < 1 + 1/e$ Radius $= \rho = 1/e$ (1)

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ (1) [Mark : 3]

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$

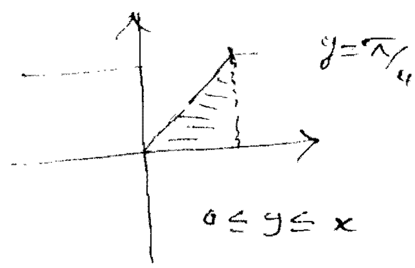
$\frac{\sin(x^2)}{x^2} = 1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \cdots$ (1)

$\int_0^{0.1} \frac{\sin(x^2)}{x^2}\,dx = \int_0^{0.1} \left[1 - \frac{x^4}{6} + \frac{x^8}{(120)8} - \cdots\right] dx$

$= \left[x - \frac{x^5}{30} + \frac{x^8}{(120)(54)} \cdots\right]_0^{0.1}$ (1)

Q; 2 (a) $\int_0^{\pi/4}\int_y \frac{x \sin x}{x^2+y^2}\,dx\,dy$ (2) $0 \le y \le \pi/4$ [Mark : 3]

$y \le x \le \pi/4$



$= \int_0^{\pi/4}\int_0^{x} x \sin x \left[\frac{1}{x^2+y^2}\right] dy\,dx$

$0 \le y \le x$

$0 \le x \le \pi/4$

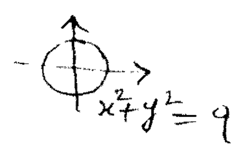
$= \int_0^{\pi/4} x \sin x \left[\frac{1}{x}\tan^{-1}(y/x)\right]_0^x dx$

$= \int_0^{\pi/4} x \sin x \left[\frac{1}{x}\tan^{-1}(1) - \frac{1}{x}\tan^{-1}(0)\right] dx = \frac{\pi}{4}\int_0 \sin x\,dx$

$= -\cos x\Big]_0^{\pi/4} = \frac{\pi}{4}\left(-\frac{1}{\sqrt{2}} + 1\right)$ (1) #

(b) $z = \sqrt{25 - x^2 - y^2} = f(x,y) \Rightarrow f_x = \frac{-x}{\sqrt{25-x^2-y^2}}, \; f_y = \frac{-y}{\sqrt{25-x^2-y^2}}$ [Mark : 3]

$S.A. = \iint_{R_{xy}} \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA = \iint_{R_{xy}} \frac{5}{\sqrt{25-x^2-y^2}}\,dA$ (1)



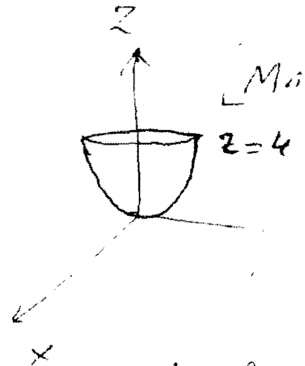
(1) $= \int_0^{2\pi}\int_0^{3} \frac{5}{\sqrt{25-r^2}}\,r\,dr\,d\theta = -\frac{5}{2}\int_0^{2\pi}\int_0^{3} (25-r^2)^{-1/2}(-2r)\,dr\,d\theta$

$= -\frac{5}{2}\int_0^{2\pi} \left[\frac{(25-r^2)^{1/2}}{1/2}\right]_0^3 d\theta = -5\int_0^{2\pi} (-1)\,d\theta = 10\pi$

(3)

(c) $m = \iiint_Q \delta \, dV = \iiint_Q (x^2+y^2)\, dV$

$= \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} (x^2+y^2)\, r \, dy \, dr \, d\theta$



[Mark · 3]

Cylindrical Shell

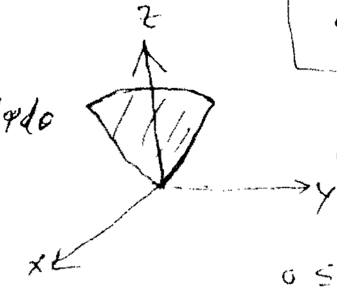
$r^2 \le \quad \le 4$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

[Mark · 3]

(d) $V = \iiint_Q dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$



(2)

$= \int_0^{2\pi}\int_0^{\pi/6} \left.\frac{\rho^3}{3}\right]_0^2 \sin\varphi \, d\varphi\, d\theta$

$0 \le \rho \le 2$

$0 \le \varphi \le \pi/6$

$0 \le \theta \le 2\pi$

$= \frac{8}{3}\int_0^{2\pi} \left. -\cos\varphi \right]_0^{\pi/6} d\theta = \frac{8}{3}\left[-\frac{\sqrt{3}}{2}+1\right]2\pi = \frac{16\pi}{3}\left(1-\frac{\sqrt{3}}{2}\right)$

(1)

Cylindrical Shell

$\sqrt{3}\, r \le z \le \sqrt{4-r^2}$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$V = \int_0^{2\pi}\int_0^{1}\int_{\sqrt{3}r}^{\sqrt{4-r^2}} r \, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{1} r\left[\sqrt{4-r^2} - \sqrt{3}\, r\right] dr\, d\theta$

(1)

$= \int_0^{2\pi}\int_0^{1}\left[\sqrt{4-r^2}\,(r) - \sqrt{3}\, r^2\right] dr\, d\theta$

$= \int_0^{2\pi}\left[-\frac{1}{2}\frac{(4-r^2)^{3/2}}{3/2} - \sqrt{3}\,\frac{r^3}{3}\right]_0^1 d\theta = \int_0^{2\pi}\left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^3}{\sqrt{3}}\right]_0^1 d\theta$

$= \left(\frac{8}{3} - \frac{4}{\sqrt{3}}\right)2\pi = \frac{16}{3}\pi - \frac{8\pi}{\sqrt{3}}$ #

(1)

Q: 3 (a) $\displaystyle\int_{(-1,2)}^{(2,3)} (2xy-3)\,dx + (x^2+4y^3+5)\,dy$ [Mark 4]

$f_x = M = 2xy - 3 \Rightarrow \partial M/\partial y = 2x$

$f_y = N = x^2+4y^3+5 \Rightarrow \dfrac{\partial N}{\partial x} = 2x$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$f_x = 2xy - 3 \Longrightarrow f(x,y) = x^2y - 3x + c_1$ (2)

$f_y = x^2+4y^3+5 \Longrightarrow f(x,y) = x^2y + y^4 + 5y + c_2$

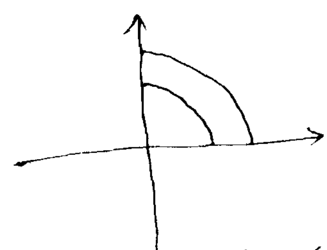
$\Rightarrow \Big[f(x,y)\Big]_{(-1,2)}^{(2,3)} = \Big[x^2y + y^4 - 3x + 5y + c_2\Big]_{(-1,2)}^{(2,3)}$

$= \Big[(4)(3) + (3)^4 - 6 + 15\Big] - \Big[2 + 16 + 3 + 10\Big] = 61$

(2)

(b) $\displaystyle\oint_C (e^x + 6xy)\,dx + (8x^2 + \sin y^2)\,dy$ [Mark 4]

($e^x + 6xy \rightarrow M$, $8x^2 + \sin y^2 \rightarrow N$)



$1 \le r \le 3$

$0 \le \theta \le \pi/2$

$= \displaystyle\iint_R (16x - 6x)\,dA$ (1) $\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$

$= \displaystyle\int_0^{\pi/2}\int_1^3 (10\,r\cos\theta)\,r\,dr\,d\theta$ (2)

$= 10\displaystyle\int_0^{\pi/2} \cos\theta \left[\dfrac{r^3}{3}\right]_1^3 d\theta =$

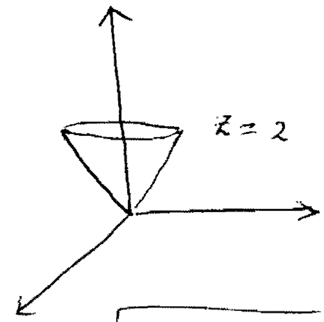
$= 10\left(\dfrac{26}{3}\right)\Big[\sin\theta\Big]_0^{\pi/2} = \dfrac{260}{3}$ #

(1)

(c) $M = xy$, $N = yz$, $P = 2x$ [Mark: 4]

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = y + z + x$$

$$= \iiint_Q (y + z + x)\, dV$$



$$= \int_0^{2\pi}\int_0^2\int_0^2 (r\sin\theta + z + r\cos\theta)\, r\, dz\, dr\, d\theta$$ (3)

$0 \le z \le 2$
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi}\int_0^2\int_0^2 (r^2\sin\theta + zr + r^2\cos\theta)\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[r^2 z \sin\theta + \frac{z^2}{2} r + r^2 z \cos\theta\right]_0^2 dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[2r^2\sin\theta + 2r + 2r^2\cos\theta\right] dr\, d\theta$$

$$= 2\int_0^{2\pi} \left[\frac{r^3}{3}\sin\theta + \frac{r^2}{2} + \frac{r^3}{3}\cos\theta\right]_0^2 d\theta$$

$$= 2\int_0^{2\pi} \left[\frac{8}{3}\sin\theta + 2 + \frac{8}{3}\cos\theta\right] d\theta$$

$$= 2\left[-\frac{8}{3}\cos\theta\right]_0^{2\pi} + 2(2\pi) + \left[\frac{8}{3}\sin\theta\right]_0^{2\pi}$$

$$= -\frac{16}{3}[1 - 1] + 4\pi + 0 = 4\pi$$

(1)

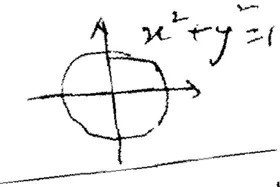
(6)

(d) $\oint_C \bar{F} \cdot d\bar{r}$

$\bar{F} = \langle y, -x, z \rangle$

$d\bar{r} = \langle dx, dy, dz \rangle$

[Mark: 4]



$$= \oint_C y\,dx - x\,dy + z\,dz$$

$C: x = \cos t,\ y = \sin t$

$z = 0, \quad 0 \le t \le 2\pi$

$dx = -\sin t\,dt$ (2)

$dy = \cos t\,dt$

$dz = 0$

$$= \int_0^{2\pi} (\sin t)(-\sin t\,dt) - (\cos t)(\cos t\,dt) + 0 \quad (1)$$

$$= -\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= -\int_0^{2\pi} dt = -2\pi \quad (1)$$

Q1

(a) Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n}{10^2 + 2^n}$

solution:

Using nth term test

$$\lim_{n\to\infty} \frac{3^n}{10^2 + 2^n} \quad \text{is} \quad \frac{\infty}{\infty}$$

By L'Hopital rule
$$= \lim_{n\to\infty} \frac{3^n \cdot \ln 3}{2^n \cdot \ln 2}$$

$$= \lim_{n\to\infty} \left(\frac{3}{2}\right)^n \cdot \frac{\ln 3}{\ln 2} = \infty$$

The series is divergent

(b) Determine whether the series $\sum_{n=1}^{\infty} n e^{-n^2}$ converges or diverges

solution:

* Applying integral test $\quad f(x) = xe^{-x^2}$

1. $f(n) = a_n \quad$ for $\quad n = 1, 2, 3, \ldots\ldots$

2. $f(x) = xe^{-x^2}$ is continuous on $[1, \infty]$

3. $f'(x) = (1)\mathrm{e}^{-x^2} + x \cdot -2xe^{x^2} = e^{-x^2}(1 - 2x^2) < 0 \quad$ for $x \geq 1$

$(f(x)$ is decreasing $)$

* integral test

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{t\to\infty} \frac{1}{-2} \int_1^t (-2)x \; e^{-x^2} dx$$

$$= \lim_{t\to\infty} \frac{-1}{2} \left[e^{-x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \frac{-1}{2}\left[e^{-t^2} - e^{-1}\right] = \frac{-1}{2}\left[0 - \frac{1}{e}\right] = \frac{1}{2e} \quad \text{convergent}$$

then the series $\sum_{n=1}^{\infty} n e^{-n^2}$ is convergent

(d) Use the Maclaurin series for the function $f(x)=\sin x$ and use its first three non-zero terms to approximate the integral $\int_0^{0.1} \frac{\sin x^2}{x^2}dx$

solution:

$f(x)=\sin x \qquad f(0)=0$

$f'(x)=\cos x \qquad f'(x)=1$

$f''(x)=-\sin x \qquad f''(x)=0$

$f'''(x)=-\cos x \qquad f'''(0)=-1$

$f^{(4)}(x)=\sin x \qquad f^{(4)}(0)=0$

Maclaurin series $\quad f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+........+\frac{x^n}{n!}f^{(n)}(0)+....$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+....+\frac{x^{2n+1}}{(2n+1)!}+...$$

Replace x by x^2 $\quad \sin x^2 \approx x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}$

Divide by x^2 $\quad \frac{\sin x^2}{x^2}\approx 1-\frac{x^4}{3!}+\frac{x^8}{5!}$

$$\int_0^{0.1}\frac{\sin x^2}{x^2}dx \simeq \int_0^{0.1}\left(1-\frac{x^4}{3!}+\frac{x^8}{5!}\right)dx$$

$$\approx \left[x-\frac{x^5}{(3!)(5)}+\frac{x^9}{(5!)(9)}\right]_0^{0.1}=$$

(c) Find the interval of convergence and radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{e^n}{n+1}(x-1)^n$$

solution: Using absolute ratio test

$$u_n = \frac{e^n}{n+1}(x-1)^n \quad , \quad u_{n+1} = \frac{e^{n+1}}{n+2}(x-1)^{n+1} = \frac{e \cdot e^n}{n+2}(x-1)(x-1)^n$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{e \cdot e^n}{n+2}(x-1)(x-1)^n \cdot \frac{n+1}{e^n (x-1)^n}\right|$$

$$= e\,|x-1| \lim_{n\to\infty}\frac{n+1}{n+2} = e\,|x-1|$$

* The power series is convergent if $e\,|x-1| < 1$

$$|x-1| < \frac{1}{e} \quad , \quad \frac{-1}{e} < x-1 < \frac{1}{e} \quad , \quad 1-\frac{1}{e} < x < 1+\frac{1}{e}$$

* Checking convergent at $x = 1-\frac{1}{e}$

$$\sum_{n=0}^{\infty}\frac{e^n}{n+1}(x-1)^n = \sum_{n=0}^{\infty}\frac{e^n}{n+1}\left(1-\frac{1}{e}-1\right)^n = \sum_{n=0}^{\infty}\frac{e^n}{n+1}\left(-\frac{1}{e}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}$$

By Alternating series test 1. $\lim_{n\to\infty}\frac{1}{n+1} = 0$

2. $f(x) = \frac{1}{x+1}$, $f'(x) = \frac{-1}{(x+1)^2} < 0$ decreasing

The series is convergent at $x = 1-\frac{1}{e}$

* Checking convergent at $x = 1+\frac{1}{e}$

$$\sum_{n=0}^{\infty}\frac{e^n}{n+1}(x-1)^n = \sum_{n=0}^{\infty}\frac{e^n}{n+1}\left(1+\frac{1}{e}-1\right)^n = \sum_{n=0}^{\infty}\frac{e^n}{n+1}\left(\frac{1}{e}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n+1}$$

is divergent (p-series)

The series is divergent at $x = 1+\frac{1}{e}$

$\Rightarrow$ The power series is convergent on $\left[1-\frac{1}{e}, 1+\frac{1}{e}\right)$, Radius $= \frac{\left(1+\frac{1}{e}\right)-\left(1-\frac{1}{e}\right)}{2} = \frac{1}{e}$

Q2:

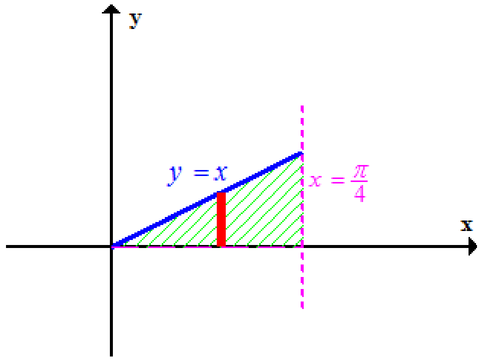
(a) Evaluate the integral $\int_0^{\pi/4}\int_y^{\pi/4} \frac{x\sin x}{x^2+y^2}\,dxdy$

solution:

$y \le x \le \frac{\pi}{4} \quad , \quad 0 \le y \le \frac{\pi}{4}$

changing to

$0 \le y \le x \quad , \quad 0 \le x \le \frac{\pi}{4}$



$$\int_0^{\pi/4}\int_y^{\pi/4} \frac{x\sin x}{x^2+y^2}\,dxdy = \int_0^{\pi/4}\int_0^{x} \frac{x\sin x}{x^2+y^2}\,dydx$$

$$= \int_0^{\pi/4}\int_0^{x} x\sin x\left[\frac{1}{x^2+y^2}\right]dydx$$

$$= \int_0^{\pi/4} x\sin x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^{\pi/4} x\sin x\cdot\frac{1}{x}\left[\tan^{-1}\left(\frac{x}{x}\right)-\tan^{-1}(0)\right]dx$$

$$= \int_0^{\pi/4} \sin x\left[\frac{\pi}{4}-0\right]dx$$

$$= \frac{\pi}{4}\left[-\cos x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\left[-\frac{\sqrt{2}}{2}+1\right]$$

(b) Find the area of the surface $x^2+y^2+z^2=25$ that lies above $z=4$

solution:

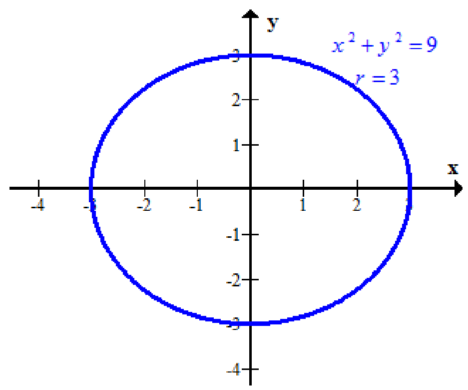
$x^2+y^2+z^2=25 \quad , \quad z^2=25-x^2-y^2$

$z=\sqrt{25-x^2-y^2} \quad , \quad f(x,y)=\sqrt{25-x^2-y^2}$

$f_x=\frac{-x}{\sqrt{25-x^2-y^2}} \quad , \quad f_y=\frac{-y}{\sqrt{25-x^2-y^2}}$

$z_1=z_2$

$\sqrt{25-x^2-y^2}=4 \quad , \quad x^2+y^2=9$



$$S.A=\iint_R \sqrt{1+[f_x]^2+[f_y]^2}\,dA$$

$$= \iint_R \sqrt{1+\frac{x^2}{25-x^2-y^2}+\frac{y^2}{25-x^2-y^2}}\,dA$$

$$= \iint_R \sqrt{\frac{25}{25-x^2-y^2}}\,dA = \iint_R \frac{5}{\sqrt{25-x^2-y^2}}\,dA$$

$$= \int_0^{2\pi}\int_0^{3} \frac{5}{\sqrt{25-r^2}}\,rdrd\theta = 5\cdot\frac{1}{-2}\int_0^{2\pi}\int_0^{3}(25-r^2)^{\frac{-1}{2}}(-2)rdrd\theta$$

$$= \frac{-5}{2}\int_0^{2\pi}\left[2(25-r^2)^{\frac{1}{2}}\right]_0^3 d\theta = -5[4-5][\theta]_0^{2\pi} = 10\pi$$

محمد ندا (أبو يوسف) 0559 108 708

(c) Set up the interated triple integral that can be used to find the mass of the solid region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ with density $\delta(x,y,z) = x^2 + y^2$

solution

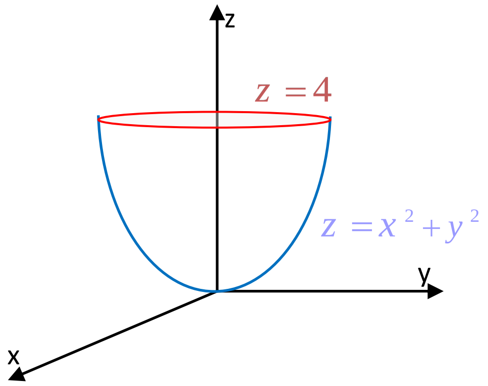
$x^2 + y^2 \le z \le 4$, $r^2 \le z \le 4$

Using cylindrical shell

$r^2 \le z \le 4$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$



$$mass = \iiint_Q \delta(x,y,z)dv$$

$$= \iiint_Q (x^2 + y^2)dv$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2 \cdot rdzdrd\theta$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^3 dzdrd\theta$$

(d) Let Q be the solid region that lies inside the cone $z = \sqrt{3x^2 + 3y^2}$ and hemi-sphere $z = \sqrt{4 - x^2 - y^2}$ Find the volume of Q

solution:

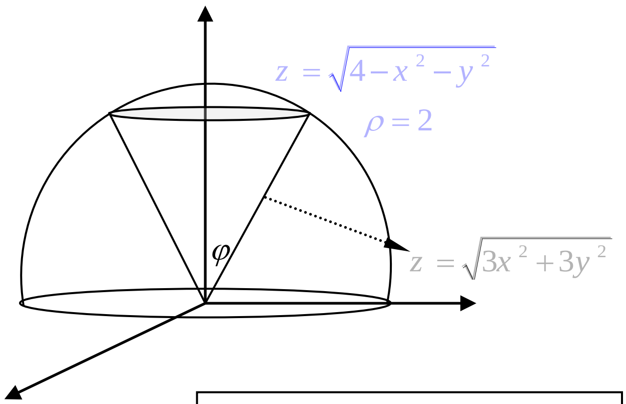
Using spherical coordinates

$\sqrt{3x^2 + 3y^2} \le z \le \sqrt{4 - x^2 - y^2}$

So $0 \le \rho \le 2$

$0 \le \varphi \le \frac{\pi}{6}$?????

$0 \le \theta \le 2\pi$



$$\text{Volume} = \iiint_Q dv$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 \rho^2 \sin\varphi d\rho d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\left[\tfrac{1}{3}\rho^3\right]_0^2 \sin\varphi d\varphi d\theta$$

$$= \frac{8}{3}\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\frac{\pi}{6}} d\theta$$

$$= \frac{8}{3}\left[-\left(\tfrac{\sqrt{3}}{2} - 1\right)\right]\left[\theta\right]_0^{2\pi} = \frac{16\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$$

* To find φ

$z = \sqrt{3x^2 + 3y^2}$

$\rho\cos\varphi = \sqrt{3(x^2 + y^2)}$

$\rho\cos\varphi = \sqrt{3\rho^2\sin^2\varphi}$

$\rho\cos\varphi = \sqrt{3}\rho\sin\varphi$

$\frac{\sin\varphi}{\cos\varphi} = \frac{1}{\sqrt{3}}$

$\tan\varphi = \frac{1}{\sqrt{3}}$, $\varphi = \frac{\pi}{6}$

Q3:

(a) Show that the line integral $\int_{(-1,2)}^{(2,3)} (2xy-3)dx+(x^2+4y^3+5)dy$ is independant of path

and find its value .

solution:

$M = 2xy - 3 \longrightarrow \frac{\partial M}{\partial y} = 2x$

$N = x^2 + 4y^3 + 5 \longrightarrow \frac{\partial N}{\partial x} = 2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the integral is independant of path

$$f(x,y) = \int M dx = \int (2xy-3)dx = x^2y - 3x + g(y)$$

$$f(x,y) = \int N dy = \int (x^2+4y^3+5)dy = x^2y + y^4 + 5y + h(x)$$

$$f(x,y) = x^2y - 3x + y^4 + 5y + c$$

$$\int_{(-1.2)}^{(2,3)} (2xy-3)dx+(x^2+4y^3+5)dy = \left[x^2y-3x+y^4+5y\right]_{(-1,2)}^{(2,3)}$$

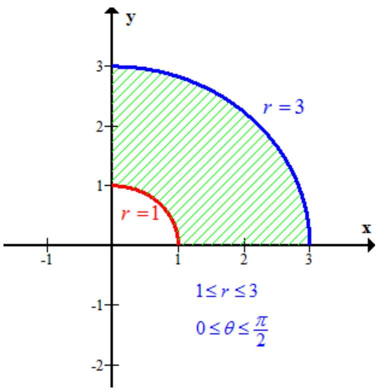
$$= \left[(4)(3)-3(2)+81+15\right]-\left[(1)(2)+3+16+10\right]$$

$$= [102]-[31] = 71$$

(b) Use Green's theorem to evaluate the line integral $\oint_C (e^x+6xy)dx+(8x^2+\sin y^2)dy$ where

C is postively oriented boundary of the region bounded by the circle of radii 1 and 3 centered at the origin and lying in the first quadrant.

solution:



$$\oint_C (e^x+6xy)dx+(8x^2+\sin y^2)dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA$$

$$= \iint_R (16x-6x)dA$$

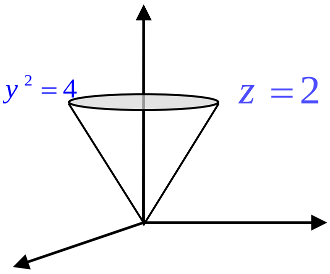
$$= \iint (10x)dA$$

$$= \int_0^{\pi/2}\int_1^3 10r\cos\theta \cdot r dr d\theta$$

$$= 10\int_0^{\frac{\pi}{2}} \left[\tfrac{1}{3}r^3\right]_1^3 \cos\theta d\theta$$

$$= 10\cdot\frac{26}{3}\left[\sin\theta\right]_0^{\frac{\pi}{2}} = \frac{260}{3}$$

(c) If S is the surface of the region bounded by the cone $z=\sqrt{x^2+y^2}$ and the planes $z=0$ and $z=2$. $\vec{F}(x,y,z)=(xy)\vec{i}+((yz)\vec{j}+(zx)\vec{k}$. use the Divergence theorem to find $\iint_S \vec{F}\bullet\vec{n}\, dS$



$0\le z\le 2$
$0\le r\le 2$
$0\le \theta\le 2\pi$

solution:

$$\iint_S \vec{F}\bullet\vec{n}\, dS=\iiint_Q \left(\tfrac{\partial M}{\partial x}+\tfrac{\partial N}{\partial y}+\tfrac{\partial P}{\partial Z}\right)dv$$

$$=\iiint_Q (y+z+x)dv$$

$$=\int_0^{2\pi}\int_0^2\int_0^2 (r\sin\theta+z+r\cos\theta)\,rdzdrd\theta$$

$$=\int_0^{2\pi}\int_0^2\int_0^2 \left(r^2\sin\theta+zr+r^2\cos\theta\right)dzdrd\theta$$

$$=\int_0^{2\pi}\int_0^2 \left(r^2\sin\theta\cdot z+\tfrac{1}{2}z^2r+r^2\cos\theta\cdot z\right)_0^2 drd\theta$$

$$=\int_0^{2\pi}\int_0^2 \left(2r^2\sin\theta+2r+2r^2\cos\theta\right)drd\theta$$

$$=\int_0^{2\pi}\left[\tfrac{2}{3}r^3\sin\theta+r^2+\tfrac{2}{3}r^3\cos\theta\right]_0^2 d\theta$$

$$=\int_0^{2\pi}\left(\tfrac{16}{3}\sin\theta+4+\tfrac{16}{3}\cos\theta\right)d\theta$$

$$=\left[-\tfrac{16}{3}\cos\theta+4\theta+\tfrac{16}{3}\sin\theta\right]_0^{2\pi}$$

$$=\left[\tfrac{-16}{3}+8\pi+0\right]-\left[\tfrac{-16}{3}+0+0\right]=8\pi$$

(d) Use the Stoke's theorem to evaluate $\iint_S (Curl\,\vec{F})\bullet\vec{n}dS$ where S is the hemi-sphere

$z=\sqrt{1-x^2-y^2}$ and the vector field is given by $\vec{F}=(y)\vec{i}-(x)\vec{j}+(z)\vec{k}$

solution:

C: $x^2+y^2=1$

$x=\cos t \quad , \quad y=\sin t \quad , \quad z=0$

$y=-\sin t dt \quad , \quad dy=\cos t dt \quad , \quad dz=0$

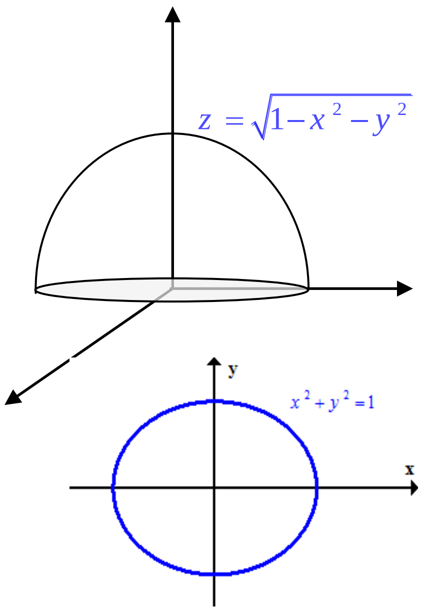
$\iint_S (Curl\,\vec{F})\bullet\vec{n}dS=\oint_C \vec{F}\cdot dr$ (line integral)

$=\oint_C ydx-xdy+zdz$

$=\int_0^{2\pi}(\sin t)(-\sin t dt)-(\cos t)(\cos t dt)+0$

$=\int_0^{2\pi}-(\sin^2 t+\cos^2 t)dt$

$=\int_0^{2\pi}-dt=-[t]_0^{2\pi}=-2\pi$



Another solution

(d) Use the Stoke's theorem to evaluate $\iint_S (Curl\,\vec{F})\bullet\vec{n}dS$ where S is the hemi-sphere

$z=\sqrt{1-x^2-y^2}$ and the vector field is given by $\vec{F}=(y)\vec{i}-(x)\vec{j}+(z)\vec{k}$

solution:

$\vec{F}\cdot dr=ydx-xdy+zdz$

$z=0 \quad , \quad dz=0$

$M=y \quad , \quad N=-x$

$\iint_S (Curl\,\vec{F})\bullet\vec{n}dS=\oint_C \vec{F}\cdot dr$ (Green's theorem)

$=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA$

$=\iint_R(-1-1)dA$

$=\int_0^{2\pi}\int_0^1-2rdrd\theta$

$=\int_0^{2\pi}\left[-r^2\right]_0^1d\theta$

$=-[\theta]_0^{2\pi}=-2\pi$