King Saud University
Department Of Mathematics.
M-203 [Final Examination]
(Differential and Integral Calculus) (I-Semester 1436/1437)

Max. Marks: 40 Time: 3 hrs

Marking Scheme: Q.No:1[2+3+4+3], Q.No:2[3+3+3+3], Q.No:3[4+4+4+4]

Q. No: 1 (a) Determine whether the series $\sum_{1}^{\infty} \frac{1}{e^n - 1}$ converges or diverges.

(b) Determine whether the series $\sum_{1}^{\infty} \frac{(-1)^n}{n + \sqrt{n}}$ converges absolutely, converges conditionally or diverges.

(c) Find the **interval of convergence** and **radius of convergence** of the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x+2)^n}{n 2^n} .$$

(d) Find the Maclaurin series for the function $f(x) = 2^x$ and use the first three terms to approximate the integral $\int_1^2 2^{-x^2} dx$.

Q. No: 2 (a) Reverse the order of integration, and evaluate the resulting integral $\int_0^1 \int_y^1 x^2 e^{xy} dx dy$.

(b) Find the surface area of the portion of the graph of $z = \sqrt{4 - x^2 - y^2}$ above the xy-plane.

(c) Find the centre of mass of the homogeneous solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = \sqrt{2}$.

(d) Find the volume of the solid bounded by the graphs of $z = x^2 + y^2$, coordinate planes, and the planes x=1, y=1 (Use a triple integral in rectangular coordinates).

Q. No: 3 (a) Use Green's theorem to evaluate $\oint_C (2xy dx + (x + y) dy$, where C is the triangle with vertices (0,0), (0,1), and (1,0).

(b) Show that the line integral $\int_{(0,0,0)}^{(1,2,1)} 3yz^2 dx + 3xz^2 dy + 6xyz dz$ is independent of path and find its value.

(c) Use the Divergence theorem to evaluate the flux $\iint_S \vec{F} \cdot \vec{n} \; dS$ of the vector force

$\vec{F}(x,y,z) = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}$ through the surface S, where S is the graph of $x^2 + y^2 + z^2 = 1$.

(d) Use Stokes' theorem to evaluate $\iint_S \left(curl \vec{F}\right) \cdot \vec{n} \; dS$, if the vector force

$\vec{F}(x,y,z) = 3z \vec{i} + 4x \vec{j} + 2y \vec{k}$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$, $z \geq 0$ and C is the trace of S in the xy-plane.

Q : 1

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n - 1}$ Comparing with $\sum \frac{1}{e^n}$ [Marks

c'gt G.S.

We can use integral test

$f(x) = \frac{1}{e^x - 1}$

(i) DEC (ii) cont. on [1, ∞)

$\displaystyle\int_1^{\infty} \frac{1}{e^x - 1}\,dx = \lim_{t \to \infty} \int_1^t \frac{e^{-x}}{1 - e^{-x}}\,dx$ c'gt

$$\lim_{n \to \infty} \frac{\frac{1}{e^n - 1}}{\frac{1}{e^n}} = \lim_{x \to \infty} \frac{e^x}{e^x - 1} = \lim_{x \to \infty} \frac{e^x}{e^x} = 1 \neq 0 \quad (1)$$

Both series c'ges or d'ge together Hence the series c'ges (1)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n + \sqrt{n}}$ Consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n + \sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

[Marks: 3]

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n \to \infty} \frac{\frac{1}{n + \sqrt{n}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{n + \sqrt{n}} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{2\sqrt{x}}}$$

$= 1$ (1)

Not A.C.

By A.S.T $\quad a_n = \frac{1}{n + \sqrt{n}}$ DEC (1)

$\lim a_n = 0 \Rightarrow$ C.C. (1)

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x+2)^n}{n\,2^n}$ [Marks. 4]

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(x+2)^{n+1}}{(n+1)\,2^{n+1}} \times \frac{n\,2^n}{(x+2)^n} \right| = \frac{n}{(n+1)2} |x+2|$$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{2} |x+2|$$

c'gt if $\frac{1}{2}|x+2| < 1 \Rightarrow |x+2| < 2$

$-2 < x+2 < 2 \Rightarrow \boxed{-4 < x < 0}$ (1)

At $x = -4$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n} = \sum_{n=1}^{\infty} -\frac{1}{n}$ d'gt

At $x = 0$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n\,2^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ c'gt

$\Rightarrow (-4, 0]$, $\rho = 2$

(d) $f(x) = 2^x \Rightarrow f(0) = 1$ [Marks 3]

$f'(x) = 2^x \ln 2 \Rightarrow f'(0) = \ln 2$

$f''(x) = 2^x (\ln 2)^2 \Rightarrow f''(0) = (\ln 2)^2$

Maclaurin Series

$$1 + x(\ln 2) + \frac{x^2}{2!}(\ln 2)^2 + \cdots + \frac{x^n}{n!}(\ln 2)^n + \cdots$$

$\Rightarrow$ Maclaurin Series of 2^{-x^2}

2

Replacing x by $(-x^2)$

$$1 - x^2(\ln 2) + \frac{x^4}{2!}(\ln 2)^2 + \cdots$$

$$\int_1^2 \left[1 - x^2(\ln 2) + \frac{x^4}{2}(\ln 2)^2\right] dx = \left. x - \frac{x^3}{3}\ln 2 + \frac{x^5}{10}(\ln 2)^2 \right]_1^2$$

$$= \left[2 - \frac{8}{3}\ln(2) + \frac{32}{10}(\ln 2)^2\right] - \left[1 - \frac{1}{3}\ln 2 + \frac{1}{10}(\ln 2)^2\right]$$

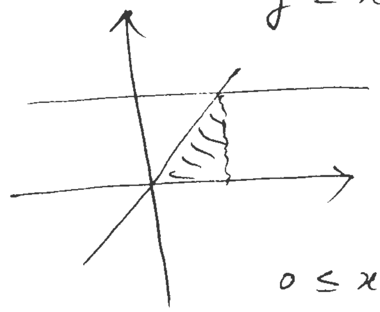
$$= 1 + \frac{7}{3}\ln(2) - \frac{31}{10}(\ln 2)^2$$

1

Q: 2 (a) $\int_0^1 \int_y^1 x^2 e^{xy}\,dx\,dy$ [Marks 3]

$0 \le y \le 1$

$y \le x \le 1$



$= \int_0^1 \int_0^x x^2 e^{xy}\,dy\,dx$ (2)

$= \int_0^1 x^2 \frac{e^{xy}}{x}\Big]_0^x dx = \int_0^1 x e^{xy}\Big]_0^x dx$

$0 \le x \le 1$

$0 \le y \le x$

$= \int_0^1 [x e^{x^2} - x]\,dx = \frac{1}{2} e^{x^2} - \frac{x^2}{2}\Big]_0^1 = \frac{1}{2} e - \frac{1}{2} - \frac{1}{2} + 0 = \frac{1}{2}e - 1$

(1)

(b) $S.A = \iint_{R_{xy}} \sqrt{1 + g_x^2 + g_y^2}\,dA$

$g_x = \frac{-x}{\sqrt{4 - x^2 - y^2}}$ [Marks 3]

$g_y = \frac{-y}{\sqrt{4 - x^2 - y^2}}$ (1)

$1 + g_x^2 + g_y^2 = 1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} = \frac{4}{4 - x^2 - y^2}$

$= \iint_{R_{xy}} \frac{2}{\sqrt{4 - x^2 - y^2}}\,dA$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$= 2\int_0^{2\pi}\int_0^2 \frac{r}{\sqrt{4 - r^2}}\,dr\,d\theta = -\frac{2}{2}\int_0^{2\pi} \frac{\sqrt{4 - r^2}}{\frac{1}{2}}\Big]_0^2 d\theta$ (1)

$= -2\,[0 - 2]\,2\pi = 8\pi$ (1)

(c) Cylindrical Coordinates [Marks 3]

$$m = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{2}} r\,dz\,dr\,d\theta \quad (2) \qquad r \le z \le \sqrt{2},\quad 0 \le r \le \sqrt{2},\quad 0 \le \theta \le 2\pi$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} [\sqrt{2} - r]\, r\, dr\, d\theta = \int_0^{2\pi} \left[\sqrt{2}\frac{r^2}{2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$$

$$= \left[\sqrt{2} - \frac{2\sqrt{2}}{3}\right] 2\pi = \frac{2\sqrt{2}\,\pi}{3}$$

For z-Coordinate

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{2}} z\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} \left[\frac{z^2}{2}\right]_r^{\sqrt{2}} r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \left[1 - \frac{r^2}{2}\right] r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{8}\right]_0^{\sqrt{2}} d\theta$$

$$= \left[1 - \frac{1}{2}\right] 2\pi = \pi$$

$$\bar{z} = \frac{\pi}{2\sqrt{2}\pi/3} = \frac{3}{2\sqrt{2}} \qquad \text{Centre } \left(0, 0, \frac{3}{2\sqrt{2}}\right) \quad (1)$$

Spherical Coordinates

$$m = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}\sec\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \qquad 0 \le \rho \le \sqrt{2}\sec\varphi,\quad 0 \le \varphi \le \pi/4,\quad 0 \le \theta \le 2\pi \quad (2)$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{(\sqrt{2}\sec\varphi)^3}{3} \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{2\sqrt{2}}{3}(\cos\varphi)^{-3} \sin\varphi\, d\varphi\, d\theta$$

$$= \frac{2\sqrt{2}}{3}\int_0^{2\pi} \left[-\frac{(\cos\varphi)^{-2}}{-2}\right]_0^{\pi/4} d\theta = \frac{2\sqrt{2}}{3}\left[+1 - \frac{1}{2}\right] 2\pi = \frac{2\sqrt{2}}{3}\pi$$

For z-coordinate

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}\sec\varphi} z\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}\sec\varphi} \rho^3 \cos\varphi \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{(\sqrt{2}\sec\varphi)^4}{4}\right] \cos\varphi \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} (\cos\varphi)^{-3} \sin\varphi\, d\varphi\, d\theta$$

$$= -\int_0^{2\pi} \left.\frac{(\cos\varphi)^{-2}}{-2}\right]_0^{\pi/4} d\theta = +\frac{1}{2}[2 - 1] 2\pi = \pi$$

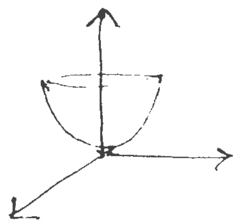
$$\Rightarrow \bar{z} = \frac{\pi}{2\sqrt{2}\pi/3} = \frac{3}{2\sqrt{2}} \Longrightarrow \text{Centre } \left(0, 0, \frac{3}{2\sqrt{2}}\right) \quad (1)$$

2(d)

$$V = \int_0^1\int_0^1\int_0^{x^2+y^2} dz\,dy\,dx$$

[Marks. 3]

2



$0 \le z \le x^2+y^2$

$0 \le x \le 1$

$0 \le y \le 1$

$$= \int_0^1\int_0^1 (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 x^2y + \frac{y^3}{3}\Big]_0^1 dx = \int_0^1 [x^2 + 1/3]\,dx = \frac{x^3}{3} + \frac{1}{3}x\Big]_0^1$$

$$= \frac{2}{3}$$

1

Q: 3(a) $\oint_C 2xy\,dx + (x+y)\,dy$

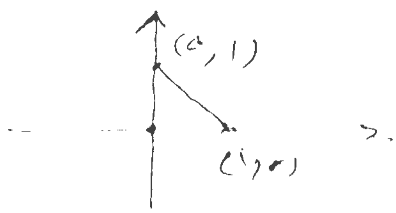
[Marks: 4]

$$= \iint_{R_{xy}} (1-2x)\,dA$$

1

$$= \int_0^1\int_0^{1-x} (1-2x)\,dy\,dx$$

2



$$= \int_0^1 \left[y - 2xy\right]_0^{1-x} dx$$

$y - 1 = \frac{-1}{1-0}(x)$

$y - 1 = -x$

$y = 1-x$

$$= \int_0^1 \left[(1-x) - 2x(1-x)\right] dx$$

$$= \int_0^1 [1 - x - 2x + 2x^2]\,dx = \int_0^1 [1 - 3x + 2x^2]\,dx$$

$$= x - \frac{3x^2}{2} + \frac{2}{3}x^3\Big]_0^1 = 1 - \frac{3}{2} + 2/3 = \frac{6-9+4}{6} = \frac{1}{6}$$

1

(b) $M = 3yz^2$, $N = 3xz^2$, $P = 6xyz$ [Marks: 4]

$$\frac{\partial M}{\partial y} = 3z^2 = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = 6zy = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = 6xz = \frac{\partial P}{\partial y}$$

$$f_x = 3yz^2 \Rightarrow f = 3xyz^2 + c_1(y,z)$$ (2)

$$f_y = 3xz^2 \Rightarrow f = 3xyz^2 + c_2(x,z)$$

$$f_z = 6xyz \Rightarrow f = 3xyz^2 + c_3(x,y)$$

$$\Big[f(x,y,z)\Big]_{(0,0,0)}^{(1,2,1)} = 3xyz^2 + c = 3(1)(2)(1)^2 - 0 = 6$$ (2)

(c) $M = x^3$, $N = y^3$, $P = z^3$ [Marks: 4]

$$\iiint_Q (3x^2 + 3y^2 + 3z^2)\, dV$$

$0 \le \rho \le 1$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4 \sin\varphi \, d\rho\, d\varphi\, d\theta$$ (3)

$$= 3\int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\Big|_0^1 \sin\varphi \, d\varphi\, d\theta = \frac{3}{5}\int_0^{2\pi}\Big[-\cos\varphi\Big]_0^{\pi} d\theta$$

$$= \frac{3}{5}[1+1]\,2\pi = \frac{12\pi}{5}$$ (1)

(1) [Marks: 4]

(d) $\oint_c M\,dx + N\,dy + P\,dz = \oint_c 3z\,dx + 4x\,dy + 2y\,dz$

$c: x = 3\cos\theta,\ y = 3\sin\theta,\ z = 0$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} 0 + 4(3\cos\theta)(3\cos\theta)\,d\theta \quad (3)$$

$$= 36\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta = 18\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 36\pi \quad (1)$$