King Saud University
Department Of Mathematics.
M-203 [Final Examination]
(Differential and Integral Calculus)
(I-Semester 1435/1436)

Max. Marks: 40 Time: 3 hrs

Marking Scheme: Q.No:1[3+3+3+3], Q.No:2[3+3+3+3], Q.No:3[4+4+4+4]

Q. No: 1 (a) Find the sum of the series $\sum_{n=1}^{\infty}\left[\frac{2}{4n^2-1}+\frac{3}{2^n}\right]$.

(b) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+1}$ is **absolutely convergent, conditionally convergent,** or **divergent.**

(c) Find the **interval of convergence** and **radius of convergence** of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.

(d) Use a power series to approximate $\int_0^1 e^{-x^2}dx$ up to four decimal places by summing up three non-zero terms of the series.

Q. No: 2 (a) Evaluate the integral $\int_0^1\int_{y^{1/3}}^{1}\sqrt[3]{y}\,3^{x^5}dx\,dy$.

(b) Find the area of the surface of the portion of the hemi-sphere $z=\sqrt{4-x^2-y^2}$ that lies inside the cylinder $x^2+y^2=1$.

(c) Find the **mass** and the **moment of inertia** about the z-axis of a solid of constant density bounded by the graphs of $x^2+y^2=1$, $z=0$, and the paraboloid $z=4-x^2-y^2$.

(d) Use Spherical co-ordinates to evaluate the integral

$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} z^2r\;dz\,dr\,d\theta$$

Q. No: 3 (a) Evaluate the line integral $\int_C xy^2ds$, where

$$C: x=\sin t,\; y=\cos t,\; z=-2t,\; 0\le t\le\pi.$$

(b) Show that the line integral $\int_{(1,0,2)}^{(-2,1,3)}(6xy^3+2z^2)dx+(9x^2y^2)dy+(4xz+1)dz$ is independent of path, and find its value.

(c) Use Divergence theorem to evaluate the flux integral $\iint_S \vec{F}\cdot\vec{n}\,dS$, where $\vec{F}(x,y,z)=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$ and S is the surface of the solid Q bounded by $x^2+y^2=4$, $z=0$, and $z=3$.

(d) Use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z)=2z\,\vec{i}+3x\,\vec{j}+5y\,\vec{k}$ and C is the curve of intersection of the paraboloid $z=4-x^2-y^2$, $z\ge 0$, and the cylinder $x^2+y^2=4$, oriented upward

(1)

(1)

Q #1 (a) $\sum_{n=1}^{\infty}\left[\frac{2}{4n^2-1}+\frac{3}{2^n}\right] = \sum_{n=1}^{\infty}\frac{2}{4n^2-1} + \sum_{n=1}^{\infty}\frac{3}{2^n}$

[Mark: 3]

$$\frac{2}{(2n-1)(2n+1)} = \frac{A}{2n-1}+\frac{B}{2n+1}$$

$$2 = A(2n+1) + B(2n-1)$$

$n=-1/2 \Rightarrow 2 = B(-1-1) \Rightarrow B=-1$

$n = 1/2 \Rightarrow 2 = A(1+1) \Rightarrow A = 1$

$$\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$ (1)

$\sum_{n=1}^{\infty}\frac{3}{2^n}$ is a G.S.

$a = \frac{3}{2}$, $r = \frac{1}{2}$

(1) $S = \frac{3/2}{1-1/2} = 6$

$$S_n = \left[\frac{1}{1}-\cancel{\frac{1}{3}}\right]+\left[\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right]+\cdots+\left[\cancel{\frac{1}{2n-1}}-\frac{1}{2n+1}\right]$$

$S_n = 1-\frac{1}{2n+1} \Rightarrow \lim_{n\to\infty} S_n = 1$ (1)

$$\Rightarrow \sum_{n=1}^{\infty}\frac{2}{4n^2-1}+\sum_{n=1}^{\infty}\frac{3}{2^n} = 1+6 = 7$$ Ans

Q1(b) $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$ Comparing with $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

[Mark: 3]

$$\frac{a_n}{b_n} = \frac{\sqrt{n}/n+1}{1/\sqrt{n}} = \frac{n}{n+1}$$

$\lim_{n\to\infty}\frac{a_n}{b_n} = 1 \Rightarrow$ not A.C. (1)

AST (i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+1} = \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{1} = 0$

(ii) $f(x) = \frac{\sqrt{x}}{x+1} \Rightarrow f'(x) = \frac{(x+1)\frac{1}{2\sqrt{x}}-\sqrt{x}}{(x+1)^2}$

$= \frac{x+1-2x}{2\sqrt{x}(x+1)^2} = \frac{-x+1}{2\sqrt{x}(x+1)^2} < 0,\ x>1$ (1)

Q#1 (c) $\left|\frac{u_{n+1}}{u_n}\right| = \left| \frac{x^{2n+3}}{(2n+1)!} \times \frac{(2n-1)!}{x^{2n-1}} \right| = \frac{1}{(2n)(2n+1)} |x^4|$ [Mark: 3]

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{1}{(2n)(2n+1)} |x^n| = 0 < 1$ (2)

c'gt for all values of x

$\Rightarrow$ Interval c'gence ∞

Radius of " ∞ (1)

Q#1 (d) $\int_0^1 e^{-x^2} dx \approx \int_0^1 \left[1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots\right] dx$ (1) [Mark: 3]

$= \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$ (1)

$= \left. x - \frac{x^3}{3} + \frac{x^5}{(5)(2!)} - \cancel{\frac{x^7}{7(3!)}} - \cdots \right]_0^1 \approx 1 - \frac{1}{3} + \frac{1}{10} + \cancel{\frac{1}{4}}$

$\approx 1 - 0.33333 + 0.10000$

≈ 0.76667 (1)

Q#2 (a) $\int_0^1 \int_{y^{1/3}}^1 \sqrt[3]{y}\; 3^{x^5}\, dx\, dy$, $0 \le y \le 1$, $y^{1/3} \le x \le 1$ [Mark: 3]

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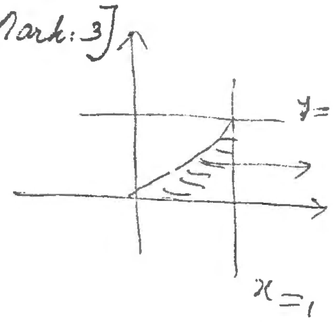

$= \int_0^1 \int_0^{x^3} y^{1/3}\, 3^{x^5}\, dy\, dx = \int_0^1 \left[\frac{y^{4/3}}{4/3} \right]_0^{x^3} dx$

$0 \le x \le 1$
$0 \le y \le x^3$

$= \frac{3}{4} \int_0^1 x^4\, 3^{x^5}\, dx$ Put $u = x^5$, $\frac{1}{5} du = x^4 dx$

$= \frac{3}{4} \int_0^1 3^u\, du = \frac{3}{4} \left[\frac{3^u}{\ln 3} \right]_0^1$

$= \frac{3}{4 \ln 3} [3 - 1] = \frac{3}{2} \frac{1}{\ln 3}$ 1

Q#2(b) $S.A. = \iint_R \sqrt{1 + [f_x]^2 + [f_y]^2}\, dA$ 1 [Mark: 3]

$f_x = \frac{-x}{\sqrt{4 - x^2 - y^2}}$

$\sqrt{1 + [f_x]^2 + [f_y]^2} = \sqrt{\frac{4}{4 - x^2 - y^2}}$ $f_y = \frac{-y}{\sqrt{4 - x^2 - y^2}}$

$S.A. = \iint_R \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA$ $x^2 + y^2 = 1$

$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$= 2 \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{4 - r^2}}\, r\, dr\, d\theta$ 1

$= \cancel{2} - \int_0^{2\pi} \int_0^1 (4 - r^2)^{-1/2} (-2r)\, dr\, d\theta = -\int_0^{2\pi} \left[\frac{(4 - r^2)^{1/2}}{1/2} \right]_0^1 d\theta$

$= -2 \left[(3)^{1/2} - 2 \right] 2\pi = 4(2 - \sqrt{3})\pi$ 1

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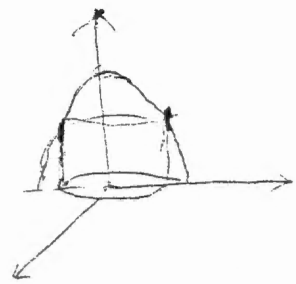
Q#2 c $m = \iiint_Q k \, dv$ [Mark:3]



$$= k \int_0^{2\pi} \int_0^{1} \int_0^{4-r^2} r \, dz \, dr \, d\theta \quad 2$$

$0 \le z \le 4 - r^2$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$= k \int_0^{2\pi} \int_0^{1} (4 - r^2) r \, dr \, d\theta$$

$$= k \int_0^{2\pi} \int_0^{1} (4r - r^3) \, dr \, d\theta = k \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4} \right]_0^1 d\theta$$

$$= k \left[2 - \frac{1}{4} \right] 2\pi = \frac{7}{2} k\pi$$

$$I_z = k \iiint_Q (x^2 + y^2) \, dv = k \int_0^{2\pi} \int_0^{1} \int_0^{4-r^2} (r^2) r \, dz \, dr \, d\theta$$

$$= k \int_0^{2\pi} \int_0^{1} (4 - r^2) r^3 \, dr \, d\theta = k \int_0^{2\pi} \left(r^4 - \frac{r^6}{6} \right)_0^1 d\theta \quad 1$$

$$= k \left(1 - \frac{1}{6} \right) 2\pi = \frac{5k\pi}{3}$$

Q#2 d $\int_0^{\pi/2} \int_0^{2} \int_0^{\sqrt{4-r^2}} z^2 \, dz \, dr \, d\theta$

$0 \le \rho \le 2$ [Mark:3]

$0 \le \varphi \le \pi/2$

$0 \le \theta \le \pi/2$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} (\rho^2 \cos^2\varphi) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \quad 2$$

$r = \rho \sin\varphi$

$z = \rho \cos\varphi$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left. \frac{\rho^5}{5} \right]_0^2 \cos^2\varphi \cdot \sin\varphi \, d\varphi \, d\theta = -\frac{32}{5} \int_0^{\pi/2} \left[\frac{\cos^3\varphi}{3} \right]_0^{\pi/2} d\theta$$

$$= -\frac{32}{5} \int_0^{\pi/2} \left[-\frac{1}{3} \right] d\theta = \frac{32}{15} \left(\frac{\pi}{2} \right) = \frac{16}{15} \pi \quad 1$$

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Q: 3 (a): [Mark: 4]

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \quad (1)$$

$$= \sqrt{\cos^2 t + \sin^2 t + 4}\ dt = \sqrt{5}\, dt \quad (1)$$

$$\int_c xy^2\, ds = \int_c^{4\pi} \sin x \cos^2 t \sqrt{5}\, dt \ (1) = \sqrt{5}\left[-\frac{\cos^3 t}{3}\right]_0^{4\pi}$$

$$= \sqrt{5}\left[-\frac{1}{3} + \frac{1}{3}\right] = \frac{2\sqrt{5}}{3} \quad (1)$$

[Mark: 4]

Q#3 (b)

$$\left.\begin{aligned} M &= 6xy^3 + 2z^2 \\ N &= 9x^2y^2 \\ P &= 4xz+1 \end{aligned}\right\} \quad \begin{aligned} \frac{\partial M}{\partial y} &= 18xy^2 = \frac{\partial N}{\partial x} \\ \frac{\partial M}{\partial z} &= 4z = \frac{\partial P}{\partial x} \\ \frac{\partial N}{\partial z} &= 0 = \frac{\partial P}{\partial y} \end{aligned}$$

$$f_x = 6xy^3 + 2z^2 \Rightarrow f(x,y,z) = 3x^2y^3 + 2xz^2 + c_1$$

$$f_y = 9x^2y^2 \Rightarrow f(x,y,z) = 3x^2y^3 + c_2 \quad (1)$$

$$f_z = 4xz + 1 \Rightarrow f(x,y,z) = 2xz^2 + z + c_3$$

$$\Rightarrow f(x,y,z) = 3x^2y^3 + 2xz^2 + z + c \quad (2)$$

$$\Rightarrow \int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2)\,dx + (9x^2y^2)\,dy + (4xz+1)\,dz$$

$$= \left[3x^2y^3 + 2xz^2 + z\right]_{(1,0,2)}^{(-2,1,3)} = \left[12 - 36 + 3\right] - \left[0 + 8 + 2\right]$$

$$= -21 - 10 = -31 \quad (1)$$

Q#(c) By D'gence theorem [Mark: 4]

$$\iint_S F\cdot\bar{n}\,ds = \iiint_G \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right) dv \quad (1)$$

$$\boxed{\begin{array}{l} M = x^3,\ N = y^3,\ P = z^3 \\ \dfrac{\partial M}{\partial x}+\dfrac{\partial N}{\partial y}+\dfrac{\partial P}{\partial z} = 3(x^2+y^2+z^2) \end{array}}$$

$$= \iiint_G 3(x^2+y^2+z^2)\,dv$$

$0 \le z \le 3$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$= 3\int_0^{2\pi}\int_0^2\int_0^3 (r^2+z^2)\,r\,dz\,dr\,d\theta \quad (2)$$

$$= 3\int_0^{2\pi}\int_0^2 \left[r^3 z + r\frac{z^3}{3}\right]_0^3 dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left[3r^3+9r\right] dr\,d\theta$$

$$= 3\int_0^{2\pi}\left[\frac{3r^4}{4}+\frac{9}{2}r^2\right]_0^2 d\theta$$

$$= 3\int_0^{2\pi}\left[12+18\right] d\theta = 90(2\pi) = 180\pi$$

(1)

[Mark : 4]

Q=3 ii) Stoke's Theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\text{Curl } \bar{F})\cdot \bar{n}\, ds$ 1

R.H.S $\text{Curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & 3x & 5y \end{vmatrix}$

$= i(5-0) - j(0-2) + k(3-0)$

$= \langle 5, 2, 3 \rangle$ 1

$\iint_S (\text{Curl } \bar{F})\cdot \bar{n}\, ds = \iint_{R_{xy}} (-M g_x - N g_y + P)\, dA$

$\boxed{\begin{matrix} M = 5 \\ N = 2 \\ P = 3 \end{matrix}\Bigg\} \quad \begin{matrix} z = g(x,y) = 4 - x^2 - y^2 \\ g_x = -2x, \; g_y = -2y \end{matrix}}$

$= \iint_{R_{xy}} (+10x + 4y + 3)\, dA$

$= \int_0^{2\pi}\int_0^2 (10r\cos\theta + 4r\sin\theta + 3)\, r\, dr\, d\theta$ 1 $\quad \boxed{\begin{matrix} \cancel{x = 2\cos t} \\ \cancel{y = 2\sin t} \\ 0 \le t \le 2\pi \end{matrix}}$

$= \int_0^{2\pi} \left[\frac{10}{3} r^3 \cos\theta + \frac{4}{3} r^3 \sin\theta + \frac{3}{2} r^2\right]_0^2 d\theta \quad \boxed{\begin{matrix} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{matrix}}$

$= \int_0^{2\pi} \left[\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right] d\theta$

$= \frac{80}{3}\sin\theta - \frac{32}{3}\cos\theta + 6\theta \Big]_0^{2\pi}$

$= -\frac{32}{3}[1-1] + 12\pi = 12\pi$ 1