از آنالیز ۱۳۹۸

ثابت کنید $(A+B)\cdot\left[(B+C)\wedge(C+A)\right] = 2A\cdot(B\wedge C)$ حسین اسدی:

$\cancel{(A+B)\cdot(U\wedge V)}$

$\cancel{D\cdot(U\wedge V)}$

$$\cancel{\underbrace{(A+B)}_{D}\cdot\underbrace{(B+C)}_{E}\wedge\underbrace{(C+A)}_{F}}$$

$$D\cdot(E\wedge F) = E\cdot(F\wedge D)$$

$$(A+B)\cdot(B\wedge C + B\wedge A + C\wedge A)$$

$$= \underline{A\cdot(B\wedge C)} + A\cdot\cancel{(B\wedge A)} + \cancel{A\cdot(C\wedge A)}$$

$$+ \cancel{B\cdot(B\wedge C)} + \cancel{B\cdot(B\wedge A)} + B\cdot(C\wedge A)$$

$$= A\cdot(B\wedge C) + B\cdot(C\wedge A)$$

$$= A\cdot(B\wedge C) + A\cdot(B\wedge C) = 2A\cdot(B\wedge C)$$

اثبات اول:

دومسئله:

$$A\wedge(B\wedge C) + B\times(C\times A) + C\times(A\times B) = 0.$$

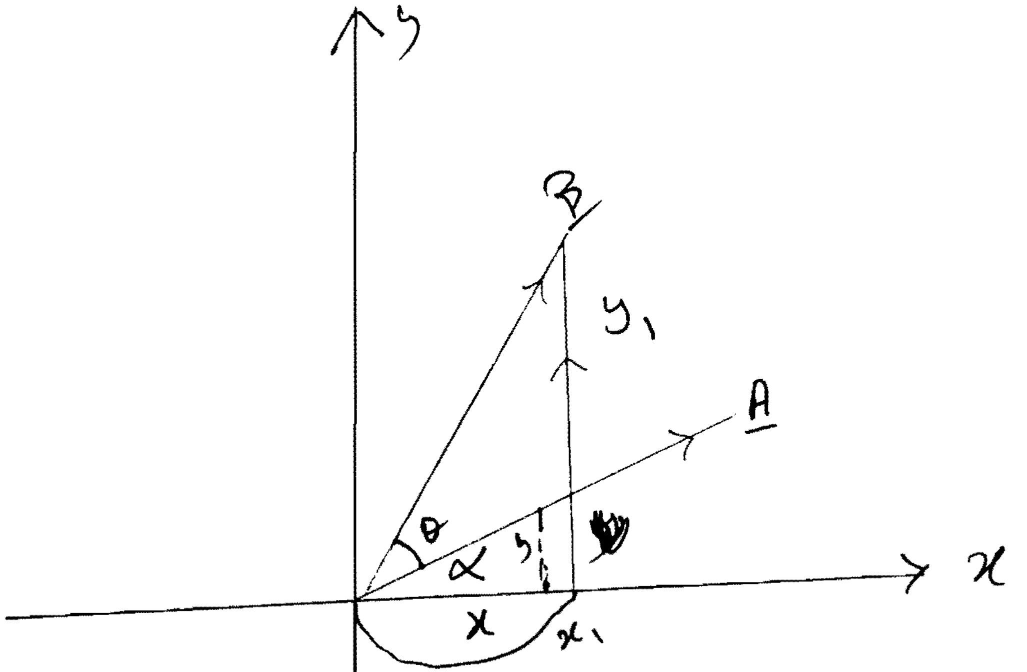
$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

$$B\times(C\wedge A) = C(B\cdot A) - A(B\cdot C)$$

$$C\times(A\times B) = A(C\cdot B) - B(C\cdot A)$$

و بکوید اکیوم جدا بیا

$$= 0$$



$$\underline{A} = x\,\underline{i} + y\,\underline{j}$$

$$\underline{B} = x_1\,\underline{i} + y_1\,\underline{j}$$

$$\underline{A}\cdot\underline{B} = x x_1 + y y_1$$

But $y_1 = |\underline{B}|\sin(\alpha+\theta)$, $\quad y = |\underline{A}|\sin\alpha$

$x_1 = |B|\cos(\alpha+\theta)$, $\quad x = |A|\cos\alpha$

$$\underline{A}\cdot\underline{B} = |A||B|\cos(\alpha+\theta)\cos\alpha + |A|\cdot|B|\sin\alpha\sin(\alpha+\theta)$$

$$\underline{A}\cdot\underline{B} = |A||B|\left[\cos(\alpha+\theta-\alpha)\right] = |A||B|\cos\theta.$$

$$\underline{A}\wedge\underline{B} = (x\,\underline{i} + y\,\underline{j})\wedge(x_1\,\underline{i} + y_1\,\underline{j})$$

$$= x y_1 k - y x_1 k = (x y_1 - y x_1)\,\underline{k}$$

$$= |A||B|\left[\sin(\alpha+\theta)\cos\alpha - \sin\alpha\cos(\alpha+\theta)\right]\underline{k}$$

$$\underline{A}\wedge\underline{B} = |\underline{A}||\underline{B}|\sin\theta\ \underline{k}$$

بين أن الانحناء للمنحنى $r = f(\theta)$ يعطى بالعلاقة :

$$K = \frac{2\left[f'(\theta)\right]^2 - f(\theta) f''(\theta) + \left(f(\theta)\right)^2}{\left[\left[f'(\theta)\right]^2 + \left[f(\theta)\right]^2\right]^{3/2}}$$

إذا كان $y = f(x)$ فإن $\underline{r}(t) = (t\,\underline{i} + f(t)\,\underline{j})$ وبالتالي فإن

$$\underline{r}'(t) = (\underline{i} + f'(t)\,\underline{j})$$

وعليه يكون $r'' = f''\,\underline{j}$ وبالتعويض عن الانحناء

$$K = \frac{r' \wedge r''}{|r'|^3} = \frac{f''}{(1 + f'^2)^{3/2}}.$$

وفي حالة الإحداثيات القطبية يكون $y = r\sin\theta$, $x = r\cos\theta$

فإن $y' = f'(x)$ وعليه

$$\frac{dy}{dx} = \frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta\, d\theta}$$

$$= \frac{\tan\theta \frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta} \quad , \quad r = f(\theta) \therefore \frac{dr}{d\theta} = f'(\theta)$$

$$\therefore \frac{dy}{dx} = \left(\frac{f'\tan\theta + f}{f' - f\tan\theta}\right) \quad , \quad \frac{d^2y}{dx^2} = f''(x)$$

$$f''(x) = \frac{d}{dx}\left(\frac{f + f'\tan\theta}{f' - f\tan\theta}\right) = \frac{d\theta}{dx}\frac{d}{d\theta}\left(\frac{f + f'\tan\theta}{f' - f\tan\theta}\right)$$

$$= \frac{d\theta}{dr} \cdot \frac{dr}{dx} \cdot \frac{d}{d\theta}\left(\frac{f + f'\tan\theta}{f' - f\tan\theta}\right)$$

$$= \frac{1}{f'(\theta)} \frac{dr}{dx} \cdot \frac{d}{d\theta}\left(\frac{f + f'\tan\theta}{f' - f\tan\theta}\right) \qquad r^2 = x^2 + y^2$$

$$\therefore \cancel{2} r\frac{dr}{dx} = \cancel{2}x + \cancel{2}y\frac{dy}{dx} \qquad \therefore \frac{dr}{dx} = \frac{x}{r} + \frac{y}{r}\frac{dy}{dx}.$$

$$\frac{dr}{dx} = \cos\theta + \sin\theta\cdot\left(\frac{f'\tan\theta + f}{f' - f\tan\theta}\right)$$

$$= \frac{f'\cos\theta - \cancel{f\sin\theta} + \cancel{f\sin\theta} + f'\sin\theta\tan\theta}{(f' - f\tan\theta)}$$

$$= f'\,\frac{1}{\cos\theta(f' - f\tan\theta)}$$

$$\therefore f''(x) = \frac{1}{\cancel{f'}}\cdot\frac{\cancel{f'}}{(f'\cos\theta - f\sin\theta)}\cdot\frac{d}{d\theta}\left(\frac{f\cos\theta + f'\sin\theta}{\cos\theta f' - f\sin\theta}\right)$$

$$= \frac{1}{(f'\cos\theta - f\sin\theta)} \times \frac{(\cos\theta f' - f\sin\theta)(f'\cos\theta - f\sin\theta + f'\cos\theta + f''\sin\theta)}{(f'\cos\theta - f\sin\theta)^2}$$

$$- \frac{(f\cos\theta + f'\sin\theta)(f''\cos\theta - f'\sin\theta - f'\sin\theta - f\cos\theta)}{}$$

$$= \frac{1}{(f'\cos\theta - f\sin\theta)^3}\Big[\cancel{f'^2\cos^2\theta} - \cancel{ff'\sin\theta\cos\theta} - \cancel{ff'\sin\theta\cos\theta} + \cancel{f^2\sin^2\theta} + \cancel{f'^2\cos^2\theta} - \cancel{ff'\sin\theta\cos\theta} + \cancel{f'f''\sin\theta\cos\theta} - \cancel{ff''\sin^2\theta}$$
$$- \cancel{ff''\cos^2\theta} - \cancel{ff'\sin\theta\cos\theta} + \cancel{ff'\sin\theta\cos\theta}$$
$$+ \cancel{f'^2\sin^2\theta} + ff'\sin\theta\cos\theta + f'^2\sin^2\theta$$
$$+ \cancel{f^2\cos^2\theta} + \cancel{ff'\sin\theta\cos\theta}\Big]$$

$$= \frac{1}{(f'\cos\theta - f\sin\theta)^3}\left[2f'^2 - ff'' + f^2\right]$$

$$\therefore K = \frac{1}{\left[1 + \left(\frac{f\cos\theta + f'\sin\theta}{f'\cos\theta - f\sin\theta}\right)^2\right]^{3/2}} \times \frac{2f'^2 - ff'' + f^2}{(f'\cos\theta - f\sin\theta)^3}$$

$$K = \frac{f'^2 \cancel{-} ff' + f^2}{\ldots}$$

$$\underline{r} = f(\theta)\, e \qquad \frac{dr}{d\theta} = f'(\theta)\underline{e} + f(\theta)\frac{d\underline{e}}{d\theta}$$

$\underline{e} = c\ldots$

$$\frac{dr}{d\theta} = f'(\theta)\,\underline{e} - f(\theta)\,\hat{\underline{e}}$$

$\frac{d\underline{e}}{d\theta} = -(\ldots$

$= -\hat{\underline{e}}$

$$\therefore \frac{d^2r}{d\theta^2} = f''(\theta)\,\underline{e} - 2f'(\theta)\,\hat{\underline{e}} + f(\theta)\,\underline{e}$$

$$= (f'' + f(\theta))\,\underline{e} - 2f'(\theta)\,\hat{\underline{e}}$$

$$\therefore r' \wedge r'' = (f'(\theta)\,\underline{e} - f(\theta)\,\hat{\underline{e}}) \wedge [(f'' + f(\theta))\,e - 2f'(\theta)\,\underline{\hat{e}}$$

$$= -f(\theta)(f'' + f(\theta))(\hat{\underline{e}} \wedge \underline{e}) - 2f'^2\,\underline{e} \wedge \hat{\underline{e}}$$

$$= [2f'^2 - f^2(\theta) - f(\theta)f''](\hat{\underline{e}} \wedge \underline{e})$$

$$\underline{r} = f(\theta)\,\underline{e}$$

$$\underline{e} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$$

$$\frac{dr}{d\theta} = f'(\theta)\,\underline{e} + f(\theta)\frac{d\underline{e}}{d\theta}$$

$$\frac{d\underline{e}}{d\theta} = -(\sin\theta\,\underline{i} - \cos\theta\,\underline{j})$$

$$= -\hat{\underline{e}}$$

$$r' = f'(\theta)\,\underline{e} - f(\theta)\,\hat{\underline{e}}$$

$$\therefore \hat{\underline{e}} = (\sin\theta\,\underline{i} - \cos\theta\,\underline{j})$$

$$r'' = f''\,\underline{e} - 2f'\,\hat{\underline{e}} - f(\theta)\,\underline{e}$$

$$(\sin\theta\,\underline{i} - \cos\theta\,\underline{j}) \wedge (\cos\theta\,\underline{i} + \sin\theta\,\underline{j})$$

$$= +\cos^2\theta + \sin^2\theta = 1$$

$$r'' = (f'' - f(\theta))\,\underline{e} - 2f'\,\hat{\underline{e}}$$

$$r' \wedge r'' = (f'(\theta)\,\underline{e} - f(\theta)\,\hat{\underline{e}}) \wedge [(f'' - f(\theta))\,\underline{e} - 2f'\,\hat{\underline{e}}]$$

$$= -f(\theta)(f'' - f(\theta))\,\hat{\underline{e}} \wedge \underline{e} - 2f'^2\,\underline{e} \wedge \hat{\underline{e}}$$

$f(\theta) = (1 + \cos\theta)$

المنحنى القلبي

$$r' \wedge r'' = (2f'^2 + f^2(\theta) - f(\theta)f'')\,\underbrace{(\hat{\underline{e}} \wedge \underline{e})}_{=1}$$

$$\therefore K = \frac{[2f'^2 + f^2(\theta) - f(\theta)f'']}{(f'^2 + f^2)^{3/2}}$$

63. Find the projection of the vector $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$ on the vector $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$. *Ans.* 8/3

64. Find the projection of the vector $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$ on the line passing through the points (2,3,−1) and (−2,−4,3).
Ans. 1

65. If $\mathbf{A}=4\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{B}=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, find a unit vector perpendicular to both **A** and **B**.
Ans. $\pm(\mathbf{i}-2\mathbf{j}-2\mathbf{k})/3$

66. Find the acute angle formed by two diagonals of a cube. *Ans.* arc cos 1/3 or $70°32'$

67. Find a unit vector parallel to the xy plane and perpendicular to the vector $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$. *Ans.* $\pm(3\mathbf{i}+4\mathbf{j})/5$

68. Show that $\mathbf{A}=(2\mathbf{i}-2\mathbf{j}+\mathbf{k})/3$, $\mathbf{B}=(\mathbf{i}+2\mathbf{j}+2\mathbf{k})/3$ and $\mathbf{C}=(2\mathbf{i}+\mathbf{j}-2\mathbf{k})/3$ are mutually orthogonal unit vectors.

69. Find the work done in moving an object along a straight line from (3,2,−1) to (2,−1,4) in a force field given by $\mathbf{F}=4\mathbf{i}-3\mathbf{j}+2\mathbf{k}$. *Ans.* 15

70. Let **F** be a constant vector force field. Show that the work done in moving an object around any closed polygon in this force field is zero.

71. Prove that an angle inscribed in a semi-circle is a right angle.

72. Let $ABCD$ be a parallelogram. Prove that $\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{DA}^2 = \overline{AC}^2+\overline{BD}^2$.

73. If $ABCD$ is any quadrilateral and P and Q are the midpoints of its diagonals, prove that
$$\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{DA}^2 = \overline{AC}^2+\overline{BD}^2+4\overline{PQ}^2$$
This is a generalization of the preceding problem.

74. (*a*) Find an equation of a plane perpendicular to a given vector **A** and distant p from the origin.
(*b*) Express the equation of (*a*) in rectangular coordinates.
Ans. (*a*) $\mathbf{r}\cdot\mathbf{n}=p$, where $\mathbf{n}=\mathbf{A}/A$; (*b*) $A_1x+A_2y+A_3z = Ap$

75. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be unit vectors in the xy plane making angles α and β with the positive x-axis.
(*a*) Prove that $\mathbf{r}_1=\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j}$, $\mathbf{r}_2=\cos\beta\,\mathbf{i}+\sin\beta\,\mathbf{j}$.
(*b*) By considering $\mathbf{r}_1\cdot\mathbf{r}_2$ prove the trigonometric formulas
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta, \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$$

76. Let **a** be the position vector of a given point (x_1,y_1,z_1), and **r** the position vector of any point (x,y,z). Describe the locus of **r** if (*a*) $|\mathbf{r}-\mathbf{a}|=3$, (*b*) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{a}=0$, (*c*) $(\mathbf{r}-\mathbf{a})\cdot\mathbf{r}=0$.
Ans. (*a*) Sphere, centre at (x_1,y_1,z_1) and radius 3.
(*b*) Plane perpendicular to **a** and passing through its terminal point.
(*c*) Sphere with centre at $(x_1/2, y_1/2, z_1/2)$ and radius $\frac{1}{2}\sqrt{x_1^2+y_1^2+z_1^2}$, or a sphere with **a** as diameter.

77. Given that $\mathbf{A}=3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{B}=\mathbf{i}-2\mathbf{j}-4\mathbf{k}$ are the position vectors of points P and Q respectively.
(*a*) Find an equation for the plane passing through Q and perpendicular to line PQ.
(*b*) What is the distance from the point (−1,1,1) to the plane?
Ans. (*a*) $(\mathbf{r}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})=0$ or $2x+3y+6z=-28$; (*b*) 5

78. Evaluate each of the following:
(*a*) $2\mathbf{j}\times(3\mathbf{i}-4\mathbf{k})$, (*b*) $(\mathbf{i}+2\mathbf{j})\times\mathbf{k}$, (*c*) $(2\mathbf{i}-4\mathbf{k})\times(\mathbf{i}+2\mathbf{j})$, (*d*) $(4\mathbf{i}+\mathbf{j}-2\mathbf{k})\times(3\mathbf{i}+\mathbf{k})$, (*e*) $(2\mathbf{i}+\mathbf{j}-\mathbf{k})\times(3\mathbf{i}-2\mathbf{j}+4\mathbf{k})$.
Ans. (*a*) $-8\mathbf{i}-6\mathbf{k}$, (*b*) $2\mathbf{i}-\mathbf{j}$, (*c*) $8\mathbf{i}-4\mathbf{j}+4\mathbf{k}$, (*d*) $\mathbf{i}-10\mathbf{j}-3\mathbf{k}$, (*e*) $2\mathbf{i}-11\mathbf{j}-7\mathbf{k}$

79. If $\mathbf{A}=3\mathbf{i}-\mathbf{j}-2\mathbf{k}$ and $\mathbf{B}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$, find: (*a*) $|\mathbf{A}\times\mathbf{B}|$, (*b*) $(\mathbf{A}+2\mathbf{B})\times(2\mathbf{A}-\mathbf{B})$, (*c*) $|(\mathbf{A}+\mathbf{B})\times(\mathbf{A}-\mathbf{B})|$.
Ans. (*a*) $\sqrt{195}$, (*b*) $-25\mathbf{i}+35\mathbf{j}-55\mathbf{k}$, (*c*) $2\sqrt{195}$

80. If $\mathbf{A}=\mathbf{i}-2\mathbf{j}-3\mathbf{k}$, $\mathbf{B}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{C}=\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, find:
(*c*) $|(\mathbf{A}\times\mathbf{B})\times\mathbf{C}|$, (*c*) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$, (*e*) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{B}\times\mathbf{C})$
(*b*) $|\mathbf{A}\times(\mathbf{B}\times\mathbf{C})|$, (*d*) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, (*f*) $(\mathbf{A}\times\mathbf{B})(\mathbf{B}\cdot\mathbf{C})$
Ans. (*a*) $5\sqrt{26}$, (*b*) $3\sqrt{10}$, (*c*) −20, (*d*) −20, (*e*) $-40\mathbf{i}-20\mathbf{j}+20\mathbf{k}$, ((*f*) $35\mathbf{i}-35\mathbf{j}+35\mathbf{k}$

81. Show that if $\mathbf{A}\neq\mathbf{0}$ and both of the conditions (*a*) $\mathbf{A}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{C}$ and (*b*) $\mathbf{A}\times\mathbf{B}=\mathbf{A}\times\mathbf{C}$ hold simultaneously then $\mathbf{B}=\mathbf{C}$, but if only one of these conditions holds then $\mathbf{B}\neq\mathbf{C}$ necessarily.

82. Find the area of a parallelogram having diagonals $\mathbf{A}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{B}=\mathbf{i}-3\mathbf{j}+4\mathbf{k}$. *Ans.* $5\sqrt{3}$

83. Find the area of a triangle with vertices at (3,−1,2), (1,−1,−3) and (4,−3,1). *Ans.* $\frac{1}{2}\sqrt{165}$

84. If $\mathbf{A} = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $\mathbf{B} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, find a vector of magnitude 5 perpendicular to both **A** and **B**.
Ans. $\pm\,\frac{5\sqrt{3}}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

85. Use Problem 75 to derive the formulas

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta, \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

86. A force given by $\mathbf{F} = 3\mathbf{i}+2\mathbf{j}-4\mathbf{k}$ is applied at the point (1,−1,2). Find the moment of **F** about the point (2,−1,3). *Ans.* $2\mathbf{i}-7\mathbf{j}-2\mathbf{k}$

87. The angular velocity of a rotating rigid body about an axis of rotation is given by $\boldsymbol{\omega} = 4\mathbf{i}+\mathbf{j}-2\mathbf{k}$. Find the linear velocity of a point P on the body whose position vector relative to a point on the axis of rotation is $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$. *Ans.* $-5\mathbf{i}-8\mathbf{j}-14\mathbf{k}$

88. Simplify $(\mathbf{A}+\mathbf{B})\cdot(\mathbf{B}+\mathbf{C})\times(\mathbf{C}+\mathbf{A})$. *Ans.* $2\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$

89. Prove that

$$(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{A}\cdot\mathbf{a} & \mathbf{A}\cdot\mathbf{b} & \mathbf{A}\cdot\mathbf{c} \\ \mathbf{B}\cdot\mathbf{a} & \mathbf{B}\cdot\mathbf{b} & \mathbf{B}\cdot\mathbf{c} \\ \mathbf{C}\cdot\mathbf{a} & \mathbf{C}\cdot\mathbf{b} & \mathbf{C}\cdot\mathbf{c} \end{vmatrix}$$

90. Find the volume of the parallelepiped whose edges are represented by $\mathbf{A} = 2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $\mathbf{B} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{C} = 3\mathbf{i}-\mathbf{j}+2\mathbf{k}$. *Ans.* 7

91. If $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 0$, show that either (*a*) **A**, **B** and **C** are coplanar but no two of them are collinear, or (*b*) two of the vectors **A**, **B** and **C** are collinear, or (*c*) all of the vectors **A**, **B** and **C** are collinear.

92. Find the constant a such that the vectors $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+a\mathbf{j}+5\mathbf{k}$ are coplanar. *Ans.* $a = -4$

93. If $\mathbf{A} = x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c}$, $\mathbf{B} = x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c}$ and $\mathbf{C} = x_3\mathbf{a}+y_3\mathbf{b}+z_3\mathbf{c}$, prove that

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})$$

94. Prove that a necessary and sufficient condition that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ is $(\mathbf{A}\times\mathbf{C})\times\mathbf{B} = \mathbf{0}$. Discuss the cases where $\mathbf{A}\cdot\mathbf{B} = 0$ or $\mathbf{B}\cdot\mathbf{C} = 0$.

95. Let points P, Q and R have position vectors $\mathbf{r}_1 = 3\mathbf{i}-2\mathbf{j}-\mathbf{k}$, $\mathbf{r}_2 = \mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $\mathbf{r}_3 = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ relative to an origin O. Find the distance from P to the plane OQR. *Ans.* 3

96. Find the shortest distance from (6,−4,4) to the line joining (2,1,2) and (3,−1,4). *Ans.* 3

97. Given points $P(2,1,3)$, $Q(1,2,1)$, $R(-1,-2,-2)$ and $S(1,-4,0)$, find the shortest distance between lines PQ and RS. *Ans.* $3\sqrt{2}$

98. Prove that the perpendiculars from the vertices of a triangle to the opposite sides (extended if necessary) meet in a point (the *orthocentre* of the triangle).

99. Prove that the perpendicular bisectors of the sides of a triangle meet in a point (the *circumcentre* of the triangle).

100. Prove that $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) + (\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{D}) = 0$.

101. Let PQR be a spherical triangle whose sides p, q, r are arcs of great circles. Prove the *law of cosines for spherical triangles*,

$$\cos p = \cos q\,\cos r + \sin q\,\sin r\,\cos P$$

with analogous formulas for $\cos q$ and $\cos r$ obtained by cyclic permutation of the letters.
[Hint: Interpret both sides of the identity $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{C}) = (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{A}) - (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{A})$.]

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FILED FARE DATA EXISTS ** >*FF
VENDOR LOCATOR DATA EXISTS ** >*VL
VENDOR REMARKS DATA EXISTS ** >*VR
SERVICE INFORMATION EXISTS ** >*SI
CUSTOM CHECK RULES EXIST ** >*RU
NE-RUHT*ALTAYYAR TRAVEL GROUP REF CRIS
2. RUHM*966507854885 REF PAX
TG-T*
OTAL 10330.00

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1)

$x = \cos^3 t \quad , \quad y = \sin^3 t$

إذا كانت $\underline{r}(t)$ هي الشكل $\underline{r}(t) = \cos^3 t\,\underline{i} + \sin^3 t\,\underline{j}$

فأوجد معادلة المماس عند النقطة $\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$ بالكوبرسار

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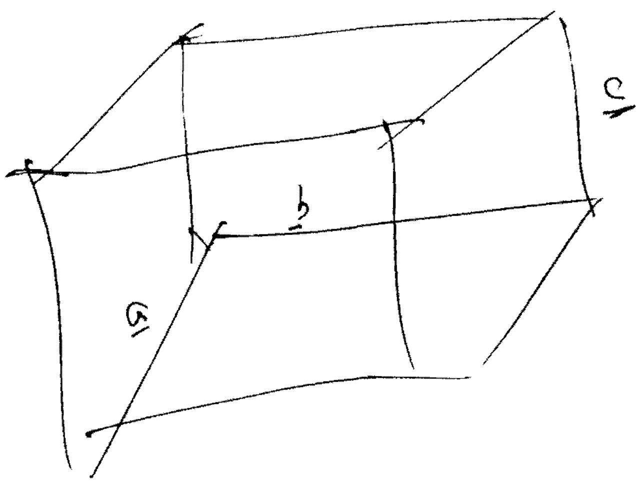
2) أوجد حجم متوازي السطوح الذي أحرفه تعطى بالمتجهات

$\underline{a} = \underline{i} + \underline{j} \quad , \quad \underline{b} = \underline{j} - \underline{k} \quad \underline{c} = \underline{k} - \underline{i}$

$\underline{c}\cdot(\underline{a}\times\underline{b})$



$\underline{a}\times\underline{b} = \underline{k} - \underline{i}$

$\underline{c} = \underline{k} - \underline{i}$

$= 1 + 1$

3) أوجد معادلة المستوى المار بالنقطة $(0,2,3)$ والعمودي على المستقيم

$\frac{x-3}{2} = \frac{y-1}{1} = \frac{z+1}{2}$

5 درجات

$x^{[illegible]} = \cos t$ $\quad x^{2/3} + y^{2/3} = 1$

$y^{2/3} = \sin^2 t$

$\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$ أوجد التقوس للمنحنى عند

$x^2 = y^2 z$

$x^2 - y^2 z = \phi$

$\frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = -2yz$

$(-2, 2, 1) \qquad \frac{\partial \phi}{\partial z} = -y^2$

$-4(x+2) - 4(y-2) - 4(z-1) = 0$

$x + 2 + y - 2 + z - 1 = 0$

$x + y + z = 1$

$K = \frac{T'}{T|r'|}$

$T = \frac{r'}{|r'|}$

$K = \frac{T'}{|T'|} =$

$\left|\frac{dT}{ds}\right| = \frac{dT}{d\theta} \cdot \frac{ds}{d}$

$= \frac{dT}{ds} \qquad 1 - 3\sin^2 t$

$T = \frac{r'}{|r'|}$

$r'' = -3[\cos^3 t - 2\sin^2 t \cos t]\, \underline{i} + 3[-\sin^3 t + 2\sin t \cos^2 t]$

$\underline{i} \quad \underline{j} \quad \underline{k}$

$r'' = -3\cos t(\cos^2 t - 2\sin^2 t)\, \underline{i} - \sin t(\sin^2 t - 2\cos^2 t)$

$= -3\cos t(1 - 3\sin^2 t)\, \underline{i} - 3\sin t(1 - 3\cos^2 t)\, \underline{j}$

$\underline{i} \quad \underline{j} \quad k$

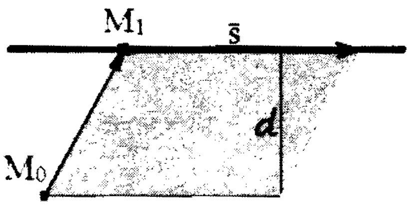
$3\sin t$

$r' = -3\sin t \cos t\, (\cos t\, \underline{i} - \sin t\, \underline{j})$

Distance from point to line - 3-Dimensional.

Distance from point to line — is equal to length of the perpendicular distance from the point to the line.

If $M_0(x_0, y_0, z_0)$ point coordinates, $\bar{s} = \{m; n; p\}$ - directing vector of line l, $M_1(x_1, y_1, z_1)$ - coordinates of point on line l, then distance between point $M_0(x_0, y_0, z_0)$ and line l can be found using the following formula



$$d = \frac{|\overline{M_0M_1} \times \bar{s}|}{|\bar{s}|}$$

Formula proving

If l is line equation then $\bar{s} = \{m; n; p\}$ - directing vector of line, $M_1(x_1, y_1, z_1)$ - coordinates of point on line.

From properties of cross product it is known that the module of cross product of vectors is equal to the area of a parallelogramme constructed on these vectors

$$S = |\overline{M_0M_1} \times \bar{s}|.$$

On the other hand parallelogramme area is equal to product of its side on height spent to this side

$$S = |\bar{s}|d.$$

Having equated the areas it is simple to receive the formula of distance from a point to a line.

Example. To find distance between point M(0, 2, 3) and line

$$\frac{x - 3}{2} = \frac{y - 1}{1} = \frac{z + 1}{2}$$

Solution.

From line equation find

$\bar{s} = \{2; 1; 2\}$ - directing vector of line;

$M_1(3; 1; -1)$ - coordinates of point on line.

Then

$$\overline{M_0M_1} = \{3 - 0; 1 - 2; -1 - 3\} = \{3; -1; -4\}$$

$$\overline{M_0M_1} \times \bar{s} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & -4 \\ 2 & 1 & 2 \end{vmatrix} =$$

$$= \boldsymbol{i}\,((-1)\cdot 2 - (-4)\cdot 1) - \boldsymbol{j}\,(3\cdot 2 - (-4)\cdot 2) + \boldsymbol{k}\,(3\cdot 1 - (-1)\cdot 2) = \{2; -14; 5\}$$

$$d = \frac{|\overline{M_0M_1} \times \bar{s}|}{|\bar{s}|} = \frac{(2^2 + (-14)^2 + 5^2)^{1/2}}{(2^2 + 1^2 + 2^2)^{1/2}} = \frac{225^{1/2}}{9^{1/2}} = 5$$

Answer: distance from point to line is equal to 5.

See also:

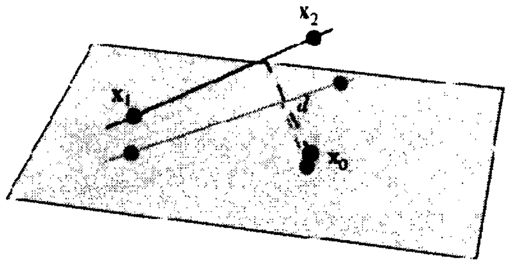
Online calculator. Distance from point to line - 3-Dimensional.

Analytic geometry.

- Distance from point to plane.
- Distance from plane to plane.
- Distance from point to line - 2-Dimensional.

Geometry > Distance >
Geometry > Line Geometry > Lines >
Interactive Entries > Interactive Demonstrations >

Point-Line Distance--3-Dimensional



Let a line in three dimensions be specified by two points $\mathbf{x}_1=(x_1,y_1,z_1)$ and $\mathbf{x}_2=(x_2,y_2,z_2)$ lying on it, so a vector along the line is given by

$$\mathbf{v}=\begin{bmatrix} x_1+(x_2-x_1)t \\ y_1+(y_2-y_1)t \\ z_1+(z_2-z_1)t \end{bmatrix}. \tag{1}$$

The squared distance between a point on the line with parameter t and a point $\mathbf{x}_0=(x_0,y_0,z_0)$ is therefore

$$d^2=[(x_1-x_0)+(x_2-x_1)t]^2+[(y_1-y_0)+(y_2-y_1)t]^2+[(z_1-z_0)+(z_2-z_1)t]^2. \tag{2}$$

To minimize the distance, set $d(d^2)/dt=0$ and solve for t to obtain

$$t=-\frac{(\mathbf{x}_1-\mathbf{x}_0)\cdot(\mathbf{x}_2-\mathbf{x}_1)}{|\mathbf{x}_2-\mathbf{x}_1|^2}, \tag{3}$$

where $\cdot$ denotes the dot product. The minimum distance can then be found by plugging t back into (2) to obtain

$$\begin{aligned} d^2 &= (x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2+ \\ &\quad 2t[(x_2-x_1)(x_1-x_0)+(y_2-y_1)(y_1-y_0)+(z_2-z_1)(z_1-z_0)]+ \\ &\quad t^2[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2] && (4) \\ &= |\mathbf{x}_1-\mathbf{x}_0|^2-2\frac{[(\mathbf{x}_1-\mathbf{x}_0)\cdot(\mathbf{x}_2-\mathbf{x}_1)]^2}{|\mathbf{x}_2-\mathbf{x}_1|^2}+\frac{[(\mathbf{x}_1-\mathbf{x}_0)\cdot(\mathbf{x}_2-\mathbf{x}_1)]^2}{|\mathbf{x}_2-\mathbf{x}_1|^2} && (5) \\ &= \frac{|\mathbf{x}_1-\mathbf{x}_0|^2|\mathbf{x}_2-\mathbf{x}_1|^2-[(\mathbf{x}_1-\mathbf{x}_0)\cdot(\mathbf{x}_2-\mathbf{x}_1)]^2}{|\mathbf{x}_2-\mathbf{x}_1|^2} && (6) \end{aligned}$$

Using the vector quadruple product

$$(\mathbf{A}\times\mathbf{B})^2=\mathbf{A}^2\,\mathbf{B}^2-(\mathbf{A}\cdot\mathbf{B})^2 \tag{7}$$

where $\times$ denotes the cross product then gives

$$d^2=\frac{|(\mathbf{x}_2-\mathbf{x}_1)\times(\mathbf{x}_1-\mathbf{x}_0)|^2}{|\mathbf{x}_2-\mathbf{x}_1|^2}, \tag{8}$$

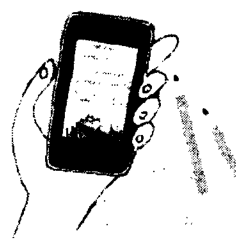
and taking the square root results in the beautiful formula

$$\begin{aligned} d &= \frac{|(\mathbf{x}_2-\mathbf{x}_1)\times(\mathbf{x}_1-\mathbf{x}_0)|}{|\mathbf{x}_2-\mathbf{x}_1|} && (9) \\ &= \frac{|(\mathbf{x}_0-\mathbf{x}_1)\times(\mathbf{x}_0-\mathbf{x}_2)|}{|\mathbf{x}_2-\mathbf{x}_1|} && (10) \\ & && (11) \end{aligned}$$

Here, the numerator is simply twice the area of the triangle formed by points $\mathbf{x}_0$, $\mathbf{x}_1$, and $\mathbf{x}_2$, and the denominator is the length of one of the bases of the triangle, which follows since, from the usual triangle area formula, $\Delta=b\,d/2$.

SEE ALSO:
Collinear, Line, Point, Point-Line Distance--2-Dimensional, Triangle Area

CITE THIS AS:
Weisstein, Eric W. "Point-Line Distance--3-Dimensional." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Point-LineDistance3-Dimensional.html



أوجد معادلة السطح الذي يمس الكرة $x^2+y^2+z^2=9$ وتقع عليه

النقطتين $(0,0,6)$ و $(0,4,0)$

لإيجاد معادلة السطح المماس

$$\nabla\phi = 2x\underline{i}+2y\underline{j}+2z\underline{k}$$

وبفرض أن (x_1,y_1,z_1) هي نقطة التماس فإن معادلة المستوى المماس للكرة هي

$$2x_1(x-x_1)+2y_1(y-y_1)+2z_1(z-z_1)=0$$

حيث $x_1^2+y_1^2+z_1^2=9$

$$\therefore \quad x_1x+y_1y+z_1z=9$$

ولكن النقطة $(0,0,6)$ تقع على السطح فهي تحقق

$$0+0+6z_1=9 \quad \therefore z_1=3/2$$

وكذلك النقطة $(0,4,0)$ فإن

$$4y_1=9 \quad \therefore y_1=9/4$$

ولكي نحصل على x_1 من

$$x_1^2+y_1^2+z_1^2=9$$

$$x_1^2+\frac{81}{16}+\frac{9}{4}=9 \qquad \therefore x_1^2=9\left(1-\tfrac{1}{4}-\frac{9}{16}\right)$$

$$=9\left(\tfrac{3}{4}-\tfrac{9}{16}\right)=\frac{27}{4}(1-$$

$$x_1=\pm\frac{\sqrt{27}}{4}=\pm\frac{3\sqrt3}{4} \Longleftarrow x_1^2=\frac{27}{16}$$

أي أن هناك مستويين مماسين للكرة أحدهما عند النقطة

$$\left(\frac{3\sqrt3}{4},\ \tfrac{9}{4},\ \tfrac{3}{2}\right)$$

والثاني عند النقطة

$$\left(-\frac{3\sqrt3}{4},\ \frac{9}{4},\ \tfrac{3}{2}\right)$$

وتكون معادلة السطح المماس

$$\frac{3\sqrt3}{4}x+\tfrac{9}{4}y+\tfrac{3}{2}z=9$$

$$\therefore\ 3\sqrt3\,x+9y+6z=18$$

$$-3\sqrt3\,x+9y+6z=18$$

$$\left(+\frac{3\sqrt{3}}{4}, \frac{9}{4}, \frac{3}{2}\right) \quad , (0,4,0), \quad (0,0,6)$$

$$A \qquad\qquad B \qquad C.$$

$$\vec{AB} = \left(-\frac{3\sqrt{3}}{4}\underline{i} + \frac{7}{4}\underline{j} - \frac{3}{2}\underline{k}\right)$$

$$\vec{BC} = \{0, -4, 6\}$$

$$\vec{AB} \wedge \vec{BC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -4 & 6 \\ -\frac{3\sqrt{3}}{4} & \frac{7}{4} & -\frac{3}{2} \end{vmatrix}$$

$$= \underline{i}\left(6 - \frac{21}{2}\right) - \underline{j}\left(\frac{18\sqrt{3}}{4}\right) + \underline{k}\; 3\sqrt{3}$$

$$= \underline{\underline{\frac{9}{2}\underline{i} - \frac{9}{2}\sqrt{3}\,\underline{j} - 3\sqrt{3}\,\underline{k}}}$$

$$\frac{9}{\cancel{2}}(x-0) - \frac{9}{\cancel{2}}\sqrt{3}\,(x-4) - 3\sqrt{3}\,(z-0) = 0$$

$$9(x) - 9\sqrt{3}\,(\cancel{x}y - 4) - 6\sqrt{3}\,z = 0$$

$$\cancel{9}\sqrt{3}(x) - \cancel{27}_{3}(y-4) - \cancel{18}_{2}z = 0$$

$$\sqrt{3}x - 3\cancel{x}y + 12 - 2z = 0$$

$$\boxed{\sqrt{3}x - 3y + 2z = 6}$$

$$\sqrt{3}x - 3y - 2z = -12$$

$$-\sqrt{3}x + 3y + 2z = 12$$

$\therefore \quad |\underline{a}\cdot\underline{b}| \leq |\underline{a}||\underline{b}| \longrightarrow (1)$

<u>Now</u>

$$|\underline{a}+\underline{b}|^2 - (|\underline{a}| + |\underline{b}|)^2$$

$$= \left[(a_1+b_1)^2 + (a_2+b_2)^2\right] - \left(\sqrt{a_1^2+a_2^2} + \sqrt{b_1^2+b_2^2}\right)^2$$

$$= \cancel{a_1^2+b_1^2+a_2^2+b_2^2} + 2a_1b_1 + 2a_2b_2 - \cancel{(a_1^2+a_2^2)} - \cancel{(b_1^2+b_2^2)}$$

$$- 2\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}$$

$$= 2\left[(a_1b_1 + a_2b_2) - \sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}\right]$$

$$= 2\left[|\underline{a}\cdot\underline{b}| - |\underline{a}||\underline{b}|\right] \longrightarrow (2)$$

$$|\underline{a}\cdot\underline{b}| - |\underline{a}||\underline{b}| \leq 0 \longrightarrow (3)$$

$$\therefore \quad |\underline{a}+\underline{b}|^2 - (|\underline{a}| + |\underline{b}|)^2 \leq 0$$

$$|\underline{a}+\underline{b}| \leq |\underline{a}| + |\underline{b}| \longrightarrow$$

بعض الإثباتات المهمة :-

(1) برهن على أن $\|a\|^2 \|b\|^2 \geq |a \cdot b|^2$

نفرض أن $a = \langle a_1, a_2 \rangle$ و $b = \langle b_1, b_2 \rangle$

ونفرض أن $(a_1 b_2 - a_2 b_1)^2 \geq 0$ أي أن

$$a_1^2 b_2^2 + a_2^2 b_1^2 \geq 2 a_1 a_2 b_1 b_2$$

وبإضافة الحد $a_1^2 b_1^2 + a_2^2 b_2^2$ إلى كلا طرفي المتباينة أي أن

$$a_1^2 b_1^2 + a_1^2 b_2 + a_2^2 b_2^2 + a_1^2 b_1^2 \geq 2 a_1 a_2 b_1 b_2 + a_1^2 b_1^2 + a_2^2 b_2^2$$

$$\therefore (a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2$$

وهذا يؤدي إلى أن

$$\|a\|^2 \|b\|^2 \geq |a \cdot b|^2 \qquad \boxed{1}$$

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(2) برهن على أن $\|a + b\| \leq \|a\| + \|b\|$

نفرض أن $X = \|a + b\|^2 - (\|a\| + \|b\|)^2$

وعليه إذا تم إثبات أن $X \leq 0$ يكون المطلوب قد تحقق

$$a + b = (a_1 + b_1) i + (a_2 + b_2) j$$

$$\therefore \|a + b\|^2 = (a_1 + b_1)^2 + (a_2 + b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2a_1 b_1 + 2a_2 b_2 \quad *$$

وحيث أن $\|a\|^2 = (a_1^2 + a_2^2)$ و $\|b\|^2 = (b_1^2 + b_2^2)$ أي أن

$$(\|a\| + \|b\|)^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) + 2\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)} \quad \#$$

$$\therefore X = 2(a_1 b_1 + a_2 b_2) - 2\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)} \quad \#$$

$$= 2\left((a \cdot b) - \|a\|\|b\|\right)$$

وباستخدام البرهان الأول $\boxed{1}$ أي أن

$$X \leq 0 \Longrightarrow \|a + b\|^2 \leq (\|a\| + \|b\|)^2 \qquad \boxed{2}$$

وهو المطلوب إثباته

تمرين (٣): $\|a+b\| \leq \|a\| + \|b\|$

نثبت أن

نفرض أن $a = \langle a_1, a_2 \rangle$ و $b = \langle b_1, b_2 \rangle$

نفرض أن $(a_1 b_2 - a_2 b_1)^2 \geq 0$

أي أن $a_1^2 b_2^2 + a_2^2 b_1^2 \geq 2a_1 a_2 b_1 b_2$

وعليه يكون $a_1^2 b_1^2 + a_2^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2 \geq 2a_1 a_2 b_1 b_2 + a_1^2 b_1^2 + a_2^2 b_2^2$

وذلك بعد إضافة $a_1^2 b_1^2 + a_2^2 b_2^2$ للطرفين المتباينة

ويكون بإعادة التجميع نحصل على

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2$$

ومنه يمكن استنتاج أن ①

$$\boxed{\|a\| \, \|b\| \geq (a \cdot b)}$$

وهذه المتباينة مهمة وسوف يتم استخدامها في الدرس القادم

والآن نحاول إثبات أن $\|a+b\| \leq \|a\| + \|b\|$

نفترض أن

$$X = \|a+b\|^2 - (\|a\| + \|b\|)^2$$

ومنه إذا تم إثبات أن $X \leq 0$ يكون المطلوب قد تحقق

$$a + b = (a_1 + b_1)\,i + (a_2 + b_2)\,j$$

$$\therefore \|a+b\|^2 = (a_1+b_1)^2 + (a_2+b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2a_1b_1 + 2b_2a_2$$

$$\|a\| = \sqrt{(a_1^2 + a_2^2)} \quad , \quad \|b\| = \sqrt{(b_1^2 + b_2^2)}$$

$$\therefore (\|a\| + \|b\|)^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) + 2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}$$

$$\therefore X = 2a_1b_1 + 2a_2b_2 - 2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}$$

وباستخدام ① نحصل على أن

$$X = 2(a \cdot b - \|a\| \, \|b\|) \leq 0$$

$$\therefore X \leq 0 \Longrightarrow \boxed{\|a+b\| \leq \|a\| + \|b\|} \quad ②$$

وهو المطلوب إثباته

① أوجد معادلة المستوى العمودي على المستوى $2x - 3y + 5z = 30$ ويمر بالنقطة $(1, -1, 2)$

② إذا كانت $\underline{A}$ و $\underline{B}$ و $\underline{C}$ ثلاث متجهات، فبرهن أن

$$(\underline{A}+\underline{B})\cdot\left[(B+C)\wedge(C+A)\right] = 2\,\underline{A}\cdot(\underline{B}\wedge\underline{C})$$

③ أوجد قيمة a والتي تجعل المتجه $A = 2a\underline{i} + 5\underline{j} + 3\underline{k}$ عمودياً على المتجه $\underline{B} = a\underline{i} + 2\underline{j} - 4a\underline{k}$

أجب عن سؤالين فقط

$$\alpha(x-1) + \beta(y+1) + \gamma(z-2) = 0$$

$$2\alpha - 3\beta + 5\gamma = 0$$

$$\gamma = 3$$
$$\beta = 1$$
$$\alpha = -1$$

$$2\alpha - 3\beta + 5 = 0$$

$$2\alpha - 3 + 15 = 0$$
$$\alpha = -\frac{12}{2} = -6$$

$$-6(x-1) + (y+1) + 3(z-2) = 0$$

$$-6x + y + 3z + \cancel{6} + 1 - \cancel{6} = 0$$

$$\boxed{6x - y - 3z = 1}$$