Complex Analysis: Problems with solutions

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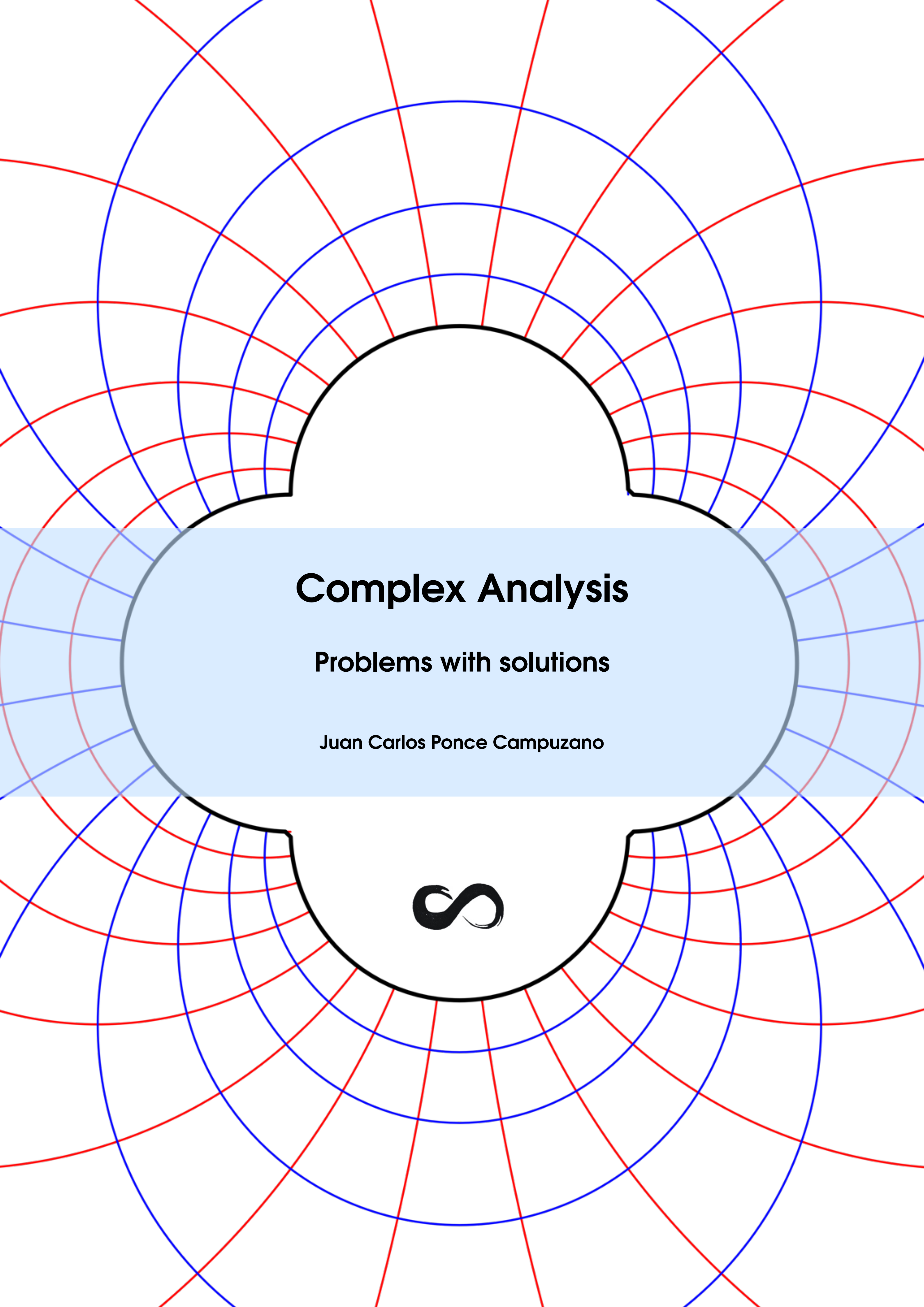

Complex Analysis

Problems with solutions

Juan Carlos Ponce Campuzano

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Foreword

This text constitutes a collection of problems for using as an additional learning resource for those who are taking an introductory course in complex analysis. The problems are numbered and allocated in four chapters corresponding to different subject areas: *Complex Numbers*, *Functions*, *Complex Integrals* and *Series*. The majority of problems are provided with answers, detailed procedures and hints (sometimes incomplete solutions).

Of course, no project such as this can be free from errors and incompleteness. I will be grateful to everyone who points out any typos, incorrect solutions, or sends any other suggestion for improving this manuscript.

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2016

1. Complex Numbers

1.1 Basic algebraic and geometric properties

1. Verify that
   (a) $\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right)=-2i$
   (b) $(2-3i)(-2+i)=-1+8i$

   Solution. We have

   $$\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right)=\sqrt{2}-i-i+\sqrt{2}=-2i,$$

   and

   $$(2-3i)(-2+i)=-4+2i+6i-3i^2=-4+3+8i=-1+8i.$$

   ■

2. Reduce the quantity

   $$\frac{5i}{(1-i)(2-i)(3-i)}$$

   to a real number.

   Solution. We have

   $$\frac{5i}{(1-i)(2-i)(3-i)}=\frac{5i}{(1-i)(5-5i)}=\frac{i}{(1-i)^2}=\frac{i}{-2i}=\frac{1}{2}$$

   ■

3. Show that
   (a) $\mathrm{Re}(iz) = -\mathrm{Im}(z)$;
   (b) $\mathrm{Im}(iz) = \mathrm{Re}(z)$.

Proof. Let $z = x + yi$ with $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Then

$$\mathrm{Re}(iz) = \mathrm{Re}(-y + xi) = -y = -\mathrm{Im}(z)$$

and

$$\mathrm{Im}(iz) = \mathrm{Im}(-y + xi) = x = \mathrm{Re}(z).$$

■

4. Verify the associative law for multiplication of complex numbers. That is, show that

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

for all $z_1, z_2, z_3 \in \mathbb{C}$.

Proof. Let $z_k = x_k + iy_k$ for $k = 1, 2, 3$. Then

$$\begin{aligned}(z_1 z_2) z_3 &= ((x_1 + y_1 i)(x_2 + y_2 i))(x_3 + y_3 i) \\ &= ((x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2))(x_3 + y_3 i) \\ &= (x_1 x_2 x_3 - x_3 y_1 y_2 - x_2 y_1 y_3 - x_1 y_2 y_3) \\ &\quad + i(x_2 x_3 y_1 + x_1 x_3 y_2 + x_1 x_2 y_3 - y_1 y_2 y_3)\end{aligned}$$

and

$$\begin{aligned}z_1 (z_2 z_3) &= (x_1 + y_1 i)((x_2 + y_2 i))(x_3 + y_3 i)) \\ &= (x_1 + y_1 i)((x_2 x_3 - y_2 y_3) + i(x_2 y_3 + x_3 y_2)) \\ &= (x_1 x_2 x_3 - x_3 y_1 y_2 - x_2 y_1 y_3 - x_1 y_2 y_3) \\ &\quad + i(x_2 x_3 y_1 + x_1 x_3 y_2 + x_1 x_2 y_3 - y_1 y_2 y_3)\end{aligned}$$

Therefore,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

■

5. Compute
   (a) $\dfrac{2+i}{2-i}$;
   (b) $(1-2i)^4$.

Answer: (a) $(3+4i)/5$, (b) $-7+24i$.

6. Let f be the map sending each complex number

$$z = x + yi \to \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Show that $f(z_1 z_2) = f(z_1) f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

Proof. Let $z_k = x_k + y_k i$ for $k = 1, 2$. Then

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2)$$

and hence

$$f(z_1 z_2) = \begin{bmatrix} x_1 x_2 - y_1 y_2 & x_2 y_1 + x_1 y_2 \\ -x_2 y_1 - x_1 y_2 & x_1 x_2 - y_1 y_2 \end{bmatrix}.$$

On the other hand,

$$f(z_1) f(z_2) = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 - y_1 y_2 & x_2 y_1 + x_1 y_2 \\ -x_2 y_1 - x_1 y_2 & x_1 x_2 - y_1 y_2 \end{bmatrix}.$$

Therefore, $f(z_1 z_2) = f(z_1) f(z_2)$. ■

7. Use binomial theorem

$$\begin{aligned}(a+b)^n &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\end{aligned}$$

to expand
(a) $(1 + \sqrt{3} i)^{2011}$;
(b) $(1 + \sqrt{3} i)^{-2011}$.

Solution. By binomial theorem,

$$(1 + \sqrt{3} i)^{2011} = \sum_{k=0}^{2011} \binom{2011}{k} (\sqrt{3} i)^k = \sum_{k=0}^{2011} \binom{2011}{k} 3^{k/2} i^k.$$

Since $i^k = (-1)^m$ for $k = 2m$ even and $i^k = (-1)^m i$ for $k = 2m+1$ odd,

$$\begin{aligned}(1 + \sqrt{3} i)^{2011} &= \sum_{0 \le 2m \le 2011} \binom{2011}{2m} 3^m (-1)^m \\ &\quad + i \sum_{0 \le 2m+1 \le 2011} \binom{2011}{2m+1} 3^m \sqrt{3} (-1)^m \\ &= \sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m + i \sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m \sqrt{3}.\end{aligned}$$

Similarly,

$$
\begin{aligned}
(1+\sqrt{3}i)^{-2011} = \left(\frac{1}{1+\sqrt{3}i}\right)^{2011} &= \left(\frac{1-\sqrt{3}i}{4}\right)^{2011} \\
&= \frac{1}{4^{2011}} \sum_{k=0}^{2011} \binom{2011}{k} (-\sqrt{3}i)^k \\
&= \frac{1}{4^{2011}} \sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m \\
&\quad - \frac{i}{4^{2011}} \sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m \sqrt{3}.
\end{aligned}
$$

■

8. Graph the following regions in the complex plane:
   (a) $\{z : \operatorname{Re} z \geq 2 \operatorname{Im} z\}$;
   (b) $\{z : \pi/2 < \operatorname{Arg} z \leq 3\pi/4\}$;
   (c) $\{z : |z-4i+2| > 2\}$.

   Solution. (a)

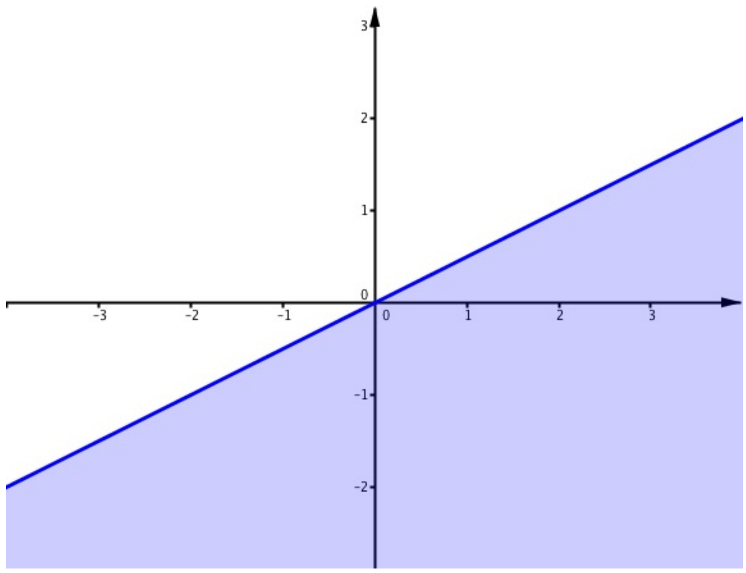

Figure 1.1:

(b)

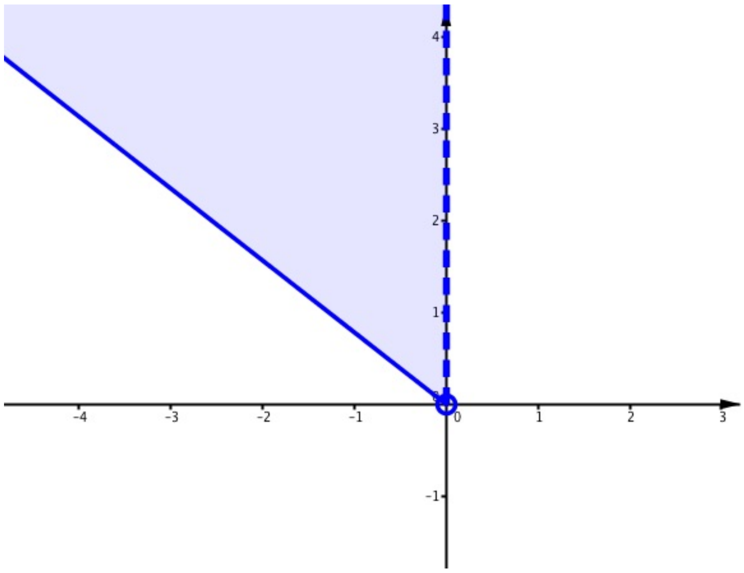

Figure 1.2:

(c)

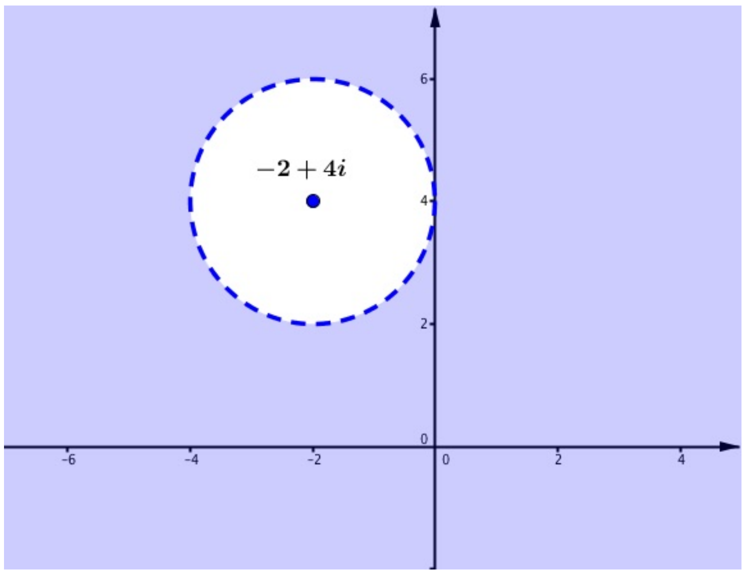

Figure 1.3:

9. Find all complex solutions of the following equations:
   (a) $\bar{z} = z$;
   (b) $\bar{z} + z = 0$;
   (c) $\bar{z} = \dfrac{9}{z}$.

   Solution. (a) Let $z = z + iy$. Thus

$$\begin{aligned} \bar{z} &= z \\ \overline{x+iy} &= x+iy \\ x-iy &= x+iy \\ -iy &= iy \\ y &= 0 \end{aligned}$$

   Hence, $\bar{z} = z$ if and only if $\operatorname{Im} z = 0$.
   (b) Let $z = z + iy$. Thus

$$\begin{aligned} \bar{z}+z &= 0 \\ \overline{x+iy}+z+iy &= 0 \\ x-iy+x+iy &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

   Hence, $\bar{z} + z$ if and only if $\operatorname{Re} z = 0$.
   (c) In this part we have

$$\bar{z} = \frac{9}{z} \iff \bar{z}z = 9 \iff |z|^2 = 9 \iff |z| = 3.$$

   Hence, $\bar{z} = \dfrac{9}{z}$ if and only if $|z| = 3$. ■

10. Suppose that z_1 and z_2 are complex numbers, with $z_1 z_2$ real and non-zero. Show that there exists a real number r such that $z_1 = r\bar{z}_2$.

   Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ with $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Thus

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + (x_1 y_2 + y_1 x_2)i$$

   Since $z_1 z_2$ is real and non-zero, $z_1 \neq 0$, $z_2 \neq 0$, and

$$x_1 x_2 - y_1 y_2 \neq 0 \quad \text{and} \quad x_1 y_2 + y_1 x_2 = 0.$$

   Thus, since $z_2 \neq 0$, then

$$\begin{aligned} \frac{z_1}{\overline{z_2}} &= \frac{x_1 + iy_1}{x_2 - iy_2} \cdot \frac{x_2 + iy_2}{x_2 + iy_2} \\ &= \frac{x_1 x_2 - y_1 y_2 + (x_1 y_2 + y_1 x_2)i}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 - y_1 y_2}{x_2^2 + y_2^2}. \end{aligned}$$

   By setting $r = \dfrac{x_1 x_2 - y_1 y_2}{x_2^2 + y_2^2}$, we have the result. ■

11. The set $\mathbb{Q}$ *adjoin* $\sqrt{2}$ is defined by $\mathbb{Q}\left(\sqrt{2}\right) = \left\{p + q\sqrt{2} : p, q \in \mathbb{Q}\right\}$.
    (a) Show that $\mathbb{Q}\left(\sqrt{2}\right)$ is a field.
    (b) Is $\sqrt{3} \in \mathbb{Q}\left(\sqrt{2}\right)$?

Proof. (a) Let $p + q\sqrt{2}, r + s\sqrt{2} \in \mathbb{Q}\left(\sqrt{2}\right)$. Since $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ is a field, we have the following:
Closure under $(+)$:

$$\left(p + q\sqrt{2}\right) + \left(r + s\sqrt{2}\right) = (p + r) + (q + s)\sqrt{2} \in \mathbb{Q}\left(\sqrt{2}\right)$$

Closure under $(\cdot)$:

$$\left(p + q\sqrt{2}\right) \cdot \left(r + s\sqrt{2}\right) = (pr + 2sq) + (rq + ps)\sqrt{2} \in \mathbb{Q}\left(\sqrt{2}\right)$$

(b) Suppose that $\sqrt{3} = a + b\sqrt{2} \in \mathbb{Q}\left(\sqrt{2}\right)$. Note that $b \neq 0$. Thus we have

$$\begin{aligned} \sqrt{3} - b\sqrt{2} &= a \\ 3 - 2\sqrt{2}\sqrt{3}b + 2b^2 &= a^2 \\ 2\sqrt{6}b &= 3 - a^2. \end{aligned}$$

Since $b \neq 0$,

$$\sqrt{6} = \frac{3 - a^2}{2b}.$$

That is, $\sqrt{6} \in \mathbb{Q}$, which is a contradiction. Therefore, $\sqrt{3} \notin \mathbb{Q}\left(\sqrt{2}\right)$ ■

1.2 Modulus

1. Show that

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

for all $z_1, z_2 \in \mathbb{C}$.

Proof. We have

$$\begin{aligned} &|z_1 - z_2|^2 + |z_1 + z_2|^2 \\ &= (z_1 - z_2)\overline{(z_1 - z_2)} + (z_1 + z_2)\overline{(z_1 + z_2)} \\ &= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) + (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= ((z_1\bar{z}_1 + z_2\bar{z}_2) - (z_1\bar{z}_2 + z_2\bar{z}_1)) + ((z_1\bar{z}_1 + z_2\bar{z}_2) + (z_1\bar{z}_2 + z_2\bar{z}_1)) \\ &= 2(z_1\bar{z}_1 + z_2\bar{z}_2) = 2(|z_1|^2 + |z_2|^2). \end{aligned}$$

■

2. Verify that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.
Hint: Reduce this inequality to $(|x|-|y|)^2 \geq 0$.

Solution. Note that

$$0 \leq (|\operatorname{Re} z| + |\operatorname{Im} z|)^2 = |\operatorname{Re} z|^2 - 2|\operatorname{Re} z||\operatorname{Im} z| + |\operatorname{Im} z|^2.$$

Thus

$$2|\operatorname{Re} z||\operatorname{Im} z| \leq |\operatorname{Re} z|^2 + |\operatorname{Im} z|^2,$$

and then

$$|\operatorname{Re} z|^2 + 2|\operatorname{Re} z||\operatorname{Im} z| + |\operatorname{Im} z|^2 \leq 2(|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2).$$

That is

$$(|\operatorname{Re} z| + |\operatorname{Im} z|)^2 \leq 2(|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2) = 2|z|^2,$$

and therefore,

$$|\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}|z|.$$

■

3. Sketch the curves in the complex plane given by
 (a) $\operatorname{Im}(z) = -1$;
 (b) $|z-1| = |z+i|$;
 (c) $2|z| = |z-2|$.

Solution. Let $z = x + yi$.
 (a) $\{\operatorname{Im}(z) = -1\} = \{y = -1\}$ is the horizontal line passing through the point $-i$.
 (b) Since

$$\begin{aligned}
|z-1| = |z+i| &\Leftrightarrow |(x-1)+yi| = |x+(y+1)i| \\
&\Leftrightarrow |(x-1)+yi|^2 = |x+(y+1)i|^2 \\
&\Leftrightarrow (x-1)^2 + y^2 = x^2 + (y+1)^2 \\
&\Leftrightarrow x + y = 0,
\end{aligned}$$

 the curve is the line $x + y = 0$.
 (c) Since

$$\begin{aligned}
2|z| = |z-2| &\Leftrightarrow 2|x+yi| = |(x-2)+yi| \\
&\Leftrightarrow 4|x+yi|^2 = |(x-2)+yi|^2 \\
&\Leftrightarrow 4(x^2+y^2) = (x-2)^2 + y^2 \\
&\Leftrightarrow 3x^2 + 4x + 3y^2 = 4 \\
&\Leftrightarrow \left(x + \frac{2}{3}\right)^2 + y^2 = \frac{16}{9} \\
&\Leftrightarrow \left|z + \frac{2}{3}\right| = \frac{4}{3}
\end{aligned}$$

 the curve is the circle with centre at $-2/3$ and radius $4/3$.

■

4. Show that
$$\frac{R^4-R}{R^2+R+1} \le \left|\frac{z^4+iz}{z^2+z+1}\right| \le \frac{R^4+R}{(R-1)^2}$$
for all z satisfying $|z| = R > 1$.

Proof. When $|z| = R > 1$,
$$|z^4+iz| \ge |z^4| - |iz| = |z|^4 - |i||z| = R^4 - R$$
and
$$|z^2+z+1| \le |z^2| + |z| + |1| = |z|^2 + |z| + 1 = R^2 + R + 1$$
by triangle inequality. Hence
$$\left|\frac{z^4+iz}{z^2+z+1}\right| \ge \frac{R^4-R}{R^2+R+1}.$$
On the other hand,
$$|z^4+iz| \le |z^4| + |iz| = |z|^4 + |i||z| = R^4 + R$$
and
$$\begin{aligned}
|z^2+z+1| &= \left|\left(z - \frac{-1+\sqrt{3}i}{2}\right)\left(z - \frac{-1-\sqrt{3}i}{2}\right)\right| \\
&= \left|z - \frac{-1+\sqrt{3}i}{2}\right|\left|z - \frac{-1-\sqrt{3}i}{2}\right| \\
&\ge \left(|z| - \left|\frac{-1+\sqrt{3}i}{2}\right|\right)\left(|z| - \left|\frac{-1-\sqrt{3}i}{2}\right|\right) \\
&= (R-1)(R-1) = (R-1)^2
\end{aligned}$$
Therefore,
$$\left|\frac{z^4+iz}{z^2+z+1}\right| \le \frac{R^4+R}{(R-1)^2}.$$
■

5. Show that
$$|\operatorname{Log}(z)| \le |\ln|z|| + \pi \qquad (1.1)$$
for all $z \ne 0$.

Proof. Since $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ for $-\pi < \operatorname{Arg}(z) \le \pi$,
$$|\operatorname{Log}(z)| = |\ln|z| + i\operatorname{Arg}(z)| \le |\ln|z|| + |i\operatorname{Arg}(z)| \le |\ln|z|| + \pi.$$
■

1.3 Exponential and Polar Form, Complex roots

1. Express the following in the form $x+iy$, with $x,y\in\mathbb{R}$:
   (a) $\dfrac{i}{1-i}+\dfrac{1-i}{i}$;
   (b) all the 3rd roots of $-8i$;
   (c) $\left(\dfrac{i+1}{\sqrt{2}}\right)^{1337}$

   Solution. (a)

$$\begin{aligned}\frac{i}{1-i}+\frac{1-i}{i} &= \frac{i^2+(1-i)^2}{(1-i)i}\\ &= \frac{-1-2i}{1-i}\cdot\frac{1-i}{1-i}\\ &= \frac{-1+i-2i-2}{2}\\ &= \frac{-3-i}{2}=-\frac{3}{2}-\frac{i}{2}\end{aligned}$$

   (b) We have that

$$-8i = 2^3\exp\left(\frac{-i\pi}{2}\right)$$

   Thus the cube roots are

$$2\exp\left(\frac{-i\pi}{6}\right),\quad 2\exp\left(\frac{i\pi}{2}\right)\quad\text{and}\quad 2\exp\left(\frac{7i\pi}{6}\right).$$

   That is

$$\sqrt{3}-i,\quad 2,\quad -\sqrt{3}-i$$

   (c)

$$\begin{aligned}\left(\frac{i+1}{\sqrt{2}}\right)^{1337} &= \left(\exp\frac{i\pi}{4}\right)^{1337}\\ &= \exp\frac{1337\pi i}{4}\\ &= \exp\left(167\cdot 2\pi i+\frac{\pi}{4}i\right)\\ &= \exp\frac{\pi}{4}i=\frac{1+i}{\sqrt{2}}.\end{aligned}$$

■

2. Find the principal argument and exponential form of
   (a) $z=\dfrac{i}{1+i}$;
   (b) $z=\sqrt{3}+i$;
   (c) $z=2-i$.

Answer:
(a) $\text{Arg}(z) = \pi/4$ and $z = (\sqrt{2}/2)\exp(\pi i/4)$.
(b) $\text{Arg}(z) = \pi/6$ and $z = 2\exp(\pi i/6)$.
(c) $\text{Arg}(z) = -\tan^{-1}(1/2)$ and $z = \sqrt{5}\exp(-\tan^{-1}(1/2)i)$.

3. Find all the complex roots of the equations:
(a) $z^6 = -9$;
(b) $z^2 + 2z + (1-i) = 0$.

Solution. (a) The roots are

$$\begin{aligned} z &= \sqrt[6]{-9} = \sqrt[6]{9e^{\pi i}} = \sqrt[3]{3}e^{\pi i/6}e^{2m\pi i/6} \ (m = 0,1,2,3,4,5) \\ &= \frac{3^{5/6}}{2} + \frac{\sqrt[3]{3}}{2}i, \sqrt[3]{3}i, -\frac{3^{5/6}}{2} + \frac{\sqrt[3]{3}}{2}i, -\frac{3^{5/6}}{2} - \frac{\sqrt[3]{3}}{2}i, -\sqrt[3]{3}i, \frac{3^{5/6}}{2} - \frac{\sqrt[3]{3}}{2}i. \end{aligned}$$

(b) The roots are

$$\begin{aligned} z &= \frac{-2 + \sqrt{4 - 4(1-i)}}{2} = -1 + \sqrt{i} \\ &= -1 + \sqrt{e^{\pi i/2}} = -1 + e^{\pi i/4}e^{2m\pi i/2} \ (m = 0,1) \\ &= \left(-1 + \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}i, \left(-1 - \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}i. \end{aligned}$$

■

4. Find the four roots of the polynomial $z^4 + 16$ and use these to factor $z^4 + 16$ into two quadratic polynomials with real coefficients.

Solution. The four roots of $z^4 + 16$ are given by

$$\begin{aligned} \sqrt[4]{-16} &= \sqrt[4]{16e^{\pi i}} = \sqrt[4]{16}e^{\pi i/4}e^{2m\pi i/4} \\ &= 2e^{\pi i/4}, 2e^{3\pi i/4}, 2e^{5\pi i/4}, 2e^{7\pi i/4} \end{aligned}$$

for $m = 0,1,2,3$. We see that these roots appear in conjugate pairs:

$$2e^{\pi i/4} = \overline{2e^{7\pi i/4}} \text{ and } 2e^{3\pi i/4} = \overline{2e^{5\pi i/4}}.$$

This gives the way to factor $z^4 + 16$ into two quadratic polynomials of real coefficients:

$$\begin{aligned} z^4 + 16 &= (z - 2e^{\pi i/4})(z - 2e^{3\pi i/4})(z - 2e^{5\pi i/4})(z - 2e^{7\pi i/4}) \\ &= \left((z - 2e^{\pi i/4})(z - 2e^{7\pi i/4})\right)\left((z - 2e^{3\pi i/4})(z - 2e^{5\pi i/4})\right) \\ &= (z^2 - 2\,\text{Re}(2e^{\pi i/4})z + 4)(z^2 - 2\,\text{Re}(2e^{3\pi i/4})z + 4) \\ &= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4) \end{aligned}$$

■

5. Do the following:
   (a) Use exponential form to compute
      i. $(1+\sqrt{3}i)^{2011}$;
      ii. $(1+\sqrt{3}i)^{-2011}$.
   (b) Prove that

$$\sum_{m=0}^{1005}\binom{2011}{2m}(-3)^m = 2^{2010}$$

and

$$\sum_{m=0}^{1005}\binom{2011}{2m+1}(-3)^m = 2^{2010}.$$

Solution. Since

$$1+\sqrt{3}i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\exp\left(\frac{\pi i}{3}\right),$$

we have

$$\begin{aligned}
(1+\sqrt{3}i)^{2011} &= 2^{2011}\exp\left(\frac{2011\pi i}{3}\right) = 2^{2011}\exp\left(\frac{2011\pi i}{3}\right)\\
&= 2^{2011}\exp\left(670\pi i+\frac{\pi i}{3}\right)\\
&= 2^{2011}\exp\left(\frac{\pi i}{3}\right) = 2^{2011}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\
&= 2^{2010}(1+\sqrt{3}i).
\end{aligned}$$

Similarly,

$$(1+\sqrt{3}i)^{-2011} = 2^{-2013}(1-\sqrt{3}i).$$

By Problem 7 in section 1.1, we have

$$\begin{aligned}
2^{2010}(1+\sqrt{3}i) &= (1+\sqrt{3}i)^{2011}\\
&= \sum_{m=0}^{1005}\binom{2011}{2m}(-3)^m + i\sum_{m=0}^{1005}\binom{2011}{2m+1}(-3)^m\sqrt{3}.
\end{aligned}$$

It follows that

$$\sum_{m=0}^{1005}\binom{2011}{2m}(-3)^m = \sum_{m=0}^{1005}\binom{2011}{2m+1}(-3)^m = 2^{2010}.$$

■

6. Establish the identity

$$1+z+z^2+\cdots+z^n = \frac{1-z^{n+1}}{1-z} \qquad (z \neq 1)$$

and then use it to derive **Lagrange's trigonometric identity**:

$$1+\cos\theta+\cos 2\theta\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}} \qquad (0<\theta<2\pi).$$

Hint: As for the first identity, write $S = 1+z+z^2+\cdots+z^n$ and consider the difference $S-zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

Proof. If $z \neq 1$, then

$$\begin{aligned}(1-z)(1+z+\cdots+z^n) &= 1+z+\cdots+z^n-(z+z^2+\cdots+z^{n+1}) \\ &= 1-z^{n+1}\end{aligned}$$

Thus

$$1+z+z^2+\cdots+z^n = \begin{cases} \dfrac{1-z^{n+1}}{1-z}, & \text{if } z \neq 1 \\ n+1, & \text{if } z=1. \end{cases}$$

Taking $z = e^{i\theta}$, where $0<\theta<2\pi$, then $z \neq 1$. Thus

$$\begin{aligned}1+e^{i\theta}+e^{2i\theta}+\cdots+e^{ni\theta} &= \frac{1-e^{(n+1)\theta}}{1-e^{i\theta}} = \frac{1-e^{(n+1)\theta}}{-e^{i\theta/2}\left(e^{i\theta/2}-e^{-i\theta/2}\right)} \\ &= \frac{-e^{-i\theta/2}(1-e^{(n+1)\theta})}{2i\sin(\theta/2)} \\ &= \frac{i\left(e^{-i\theta/2}-e^{(n+\frac{1}{2})i\theta}\right)}{2\sin(\theta/2)} \\ &= \frac{1}{2}+\frac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}+i\frac{\cos(\theta/2)-\cos[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}\end{aligned}$$

Equating real and imaginary parts, we obtain

$$1+\cos\theta+\cos 2\theta\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}$$

and

$$\sin\theta+\sin 2\theta\cdots+\sin n\theta = \frac{\cos(\theta/2)-\cos[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}.$$

■

7. Use complex numbers to prove the **Law of Cosine**: Let ΔABC be a triangle with $|BC| = a$, $|CA| = b$, $|AB| = c$ and $\angle BCA = \theta$. Then

$$a^2+b^2-2ab\cos\theta = c^2.$$

Hint: Place C at the origin, B at z_1 and A at z_2. Prove that

$$z_1\bar{z}_2+z_2\bar{z}_1 = 2|z_1z_2|\cos\theta.$$

Proof. Following the hint, we let $C = 0$, $B = z_1$ and $A = z_2$. Then $a = |z_1|$, $b = |z_2|$ and $c = |z_2 - z_1|$. So

$$\begin{aligned}
a^2 + b^2 - c^2 &= |z_1|^2 + |z_2|^2 - |z_2 - z_1|^2 \\
&= (z_1\bar{z}_1 + z_2\bar{z}_2) - (z_2 - z_1)\overline{(z_2 - z_1)} \\
&= (z_1\bar{z}_1 + z_2\bar{z}_2) - (z_2 - z_1)(\bar{z}_2 - \bar{z}_1) \\
&= (z_1\bar{z}_1 + z_2\bar{z}_2) - (z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1) \\
&= z_1\bar{z}_2 + z_2\bar{z}_1.
\end{aligned}$$

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$\begin{aligned}
z_1\bar{z}_2 + z_2\bar{z}_1 &= r_1 e^{i\theta_1}\overline{r_2 e^{i\theta_2}} + r_2 e^{i\theta_2}\overline{r_1 e^{i\theta_1}} \\
&= (r_1 e^{i\theta_1})(r_2 e^{-i\theta_2}) + (r_2 e^{i\theta_2})(r_1 e^{-i\theta_1}) \\
&= r_1 r_2 e^{i(\theta_1 - \theta_2)} + r_1 r_2 e^{i(\theta_2 - \theta_1)} \\
&= 2r_1 r_2 \cos(\theta_1 - \theta_2) = 2|z_1||z_2|\cos\theta = 2ab\cos\theta.
\end{aligned}$$

Therefore, we have

$$a^2 + b^2 - c^2 = z_1\bar{z}_2 + z_2\bar{z}_1 = 2ab\cos\theta$$

and hence

$$a^2 + b^2 - 2ab\cos\theta = c^2.$$

■

2. Functions

2.1 Basic notions

1. Write the following functions $f(z)$ in the forms $f(z) = u(x,y) + iv(x,y)$ under Cartesian coordinates with $u(x,y) = \mathrm{Re}(f(z))$ and $v(x,y) = \mathrm{Im}(f(z))$:
   (a) $f(z) = z^3 + z + 1$
   (b) $f(z) = z^3 - z$;
   (c) $f(z) = \dfrac{1}{i-z}$;
   (d) $f(z) = \overline{\exp(z^2)}$.

Solution. (a)

$$\begin{aligned} f(z) &= (x+iy)^3 + (x+iy) + 1 \\ &= (x+iy)(x^2 - y^2 + 2ixy) + x + iy + 1 \\ &= x^3 - xy^2 + 2ix^2y + ix^2y - iy^3 - 2xy^2 + x + iy + 1 \\ &= x^3 - 3xy^2 + x + 1 + i(3x^2y - y^3 + y). \end{aligned}$$

(b)

$$\begin{aligned} f(z) = z^3 - z &= (x+yi)^3 - (x+yi) \\ &= (x^3 + 3x^2yi - 3xy^2 - y^3 i) - (x+yi) \\ &= (x^3 - 3xy^2 - x) + i(3x^2y - y^3 - y), \end{aligned}$$

(c)

$$\begin{aligned} f(z) &= \frac{1}{i-z} = \frac{1}{-x+(1-y)i} \\ &= \frac{-x-(1-y)i}{x^2+(1-y)^2} \\ &= -\frac{x}{x^2+(1-y)^2} - i\frac{1-y}{x^2+(1-y)^2} \end{aligned}$$

(d)

$$\begin{aligned} f(z) &= \overline{\exp(z^2)} = \overline{\exp((x+yi)^2)} \\ &= \overline{\exp((x^2-y^2)+2xyi)} \\ &= \overline{e^{x^2-y^2}(\cos(2xy)+i\sin(2xy))} \\ &= e^{x^2-y^2}\cos(2xy) - ie^{x^2-y^2}\sin(2xy) \end{aligned}$$

■

2. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}$$

to write $f(z)$ in terms of z and simplify the result.

Solution. We have

$$\begin{aligned} f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\ &= x^2 - y^2 + i2x - i2xy - 2y \\ &= (x - iy)^2 + i(2x + 2iy) \\ &= \bar{z}^2 + 2iz. \end{aligned}$$

■

3. Suppose $p(z)$ is a polynomial with real coefficients. Prove that
   (a) $\overline{p(z)} = p(\bar{z})$;
   (b) $p(z) = 0$ if and only if $p(\bar{z}) = 0$;
   (c) the roots of $p(z) = 0$ appear in conjugate pairs, i.e., if z_0 is a root of $p(z) = 0$, so is $\bar{z}_0$.

Proof. Let $p(z) = a_0 + a_1 z + ... + a_n z^n$ for $a_0, a_1, ..., a_n \in \mathbb{R}$. Then

$$\begin{aligned} \overline{p(z)} &= \overline{a_0 + a_1 z + \cdots + a_n z^n} \\ &= \overline{a_0} + \overline{a_1 z} + \cdots + \overline{a_n z^n} \\ &= \overline{a_0} + (\overline{a_1})\bar{z} + \cdots + (\overline{a_n})\overline{z^n} \\ &= a_0 + a_1\bar{z} + \cdots + a_n\bar{z}^n = p(\bar{z}). \end{aligned}$$

If $p(z) = 0$, then $\overline{p(z)} = 0$ and hence $p(\bar{z}) = \overline{p(z)} = 0$; on the other hand, if $p(\bar{z}) = 0$, then $\overline{p(z)} = p(\bar{z}) = 0$ and hence $p(z) = 0$.
By the above, $p(z_0) = 0$ if and only if $p(\bar{z}_0) = 0$. Therefore, z_0 is a root of $p(z) = 0$ if and only if $\bar{z}_0$ is. ■

4. Let
$$T(z) = \frac{z}{z+1}.$$
Find the inverse image of the disk $|z| < 1/2$ under T and sketch it.

Solution. Let $D = \{|z| < 1/2\}$. The inverse image of D under T is
$$\begin{aligned} T^{-1}(D) &= \{z \in \mathbb{C} : T(z) \in D\} = \{|T(z)| < \frac{1}{2}\} \\ &= \left\{z : \left|\frac{z}{z+1}\right| < \frac{1}{2}\right\} = \{2|z| < |z+1|\}. \end{aligned}$$
Let $z = x + yi$. Then
$$\begin{aligned} 2|z| < |z+1| &\Leftrightarrow 4(x^2+y^2) < (x+1)^2 + y^2 \\ &\Leftrightarrow 3x^2 - 2x + 3y^2 < 1 \\ &\Leftrightarrow \left(x - \frac{1}{3}\right)^2 + y^2 < \frac{4}{9} \\ &\Leftrightarrow \left|z - \frac{1}{3}\right| < \frac{2}{3} \end{aligned}$$
So
$$T^{-1}(D) = \left\{z : \left|z - \frac{1}{3}\right| < \frac{2}{3}\right\}$$
is the disk with centre at $1/3$ and radius $2/3$. ■

5. Sketch the following sets in the complex plane $\mathbb{C}$ and determine whether they are open, closed, or neither; bounded; connected. Briefly state your reason.
   (a) $|z+3| < 1$;
   (b) $|\operatorname{Im}(z)| \geq 1$;
   (c) $1 \leq |z+3| < 2$.

Solution. (a) Since $\{|z+3| < 1\} = \{(x+3)^2 + y^2 - 1 < 0\}$ and $f(x,y) = (x+3)^2 + y^2 - 1$ is a continuous function on $\mathbb{R}^2$, the set is open. It is not closed since the only sets that are both open and closed in $\mathbb{C}$ are $\emptyset$ and $\mathbb{C}$.
Since
$$|z| = |z+3-3| \leq |z+3| + |-3| = |z+3| + 3 < 4$$
for all $|z+3| < 1$, $\{|z+3| < 1\} \subset \{|z| < 4\}$ and hence it is bounded.
It is connected since it is a convex set. ■

Solution. (b) We have
$$\{|\operatorname{Im}(z)| \geq 1\} = \{|y| \geq 1\} = \{y \geq 1\} \cup \{y \leq -1\}.$$

Since $f(x,y) = y$ is continuous on $\mathbb{R}^2$, both $\{y \geq 1\}$ and $\{y \leq -1\}$ are closed and hence $\{|\operatorname{Im}(z)| \geq 1\}$ is closed. It is not open since the only sets that are both open and closed in $\mathbb{C}$ are $\emptyset$ and $\mathbb{C}$.
Since $z_n = n + 2i \in \{|\operatorname{Im}(z)| \geq 1\}$ for all $n \in \mathbb{Z}$ and

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} \sqrt{n^2+4} = \infty,$$

the set is unbounded.
The set is not connected. Otherwise, let $p = 2i$ and $q = -2i$. There is a polygonal path

$$\overline{p_0p_1} \cup \overline{p_1p_2} \cup \ldots \cup \overline{p_{n-1}p_n}$$

with $p_0 = p$, $p_n = q$ and $p_k \in \{|\operatorname{Im}(z)| \geq 1\}$ for all $0 \leq k \leq n$.
Let $0 \leq m \leq n$ be the largest integer such that $p_m \in \{y \geq 1\}$. Then $p_{m+1} \in \{y \leq -1\}$. So $\operatorname{Im}(p_m) \geq 1 > 0$ and $\operatorname{Im}(p_{m+1}) \leq -1 < 0$. It follows that there is a point $p \in \overline{p_mp_{m+1}}$ such that $\operatorname{Im}(p) = 0$. This is a contradiction since $\overline{p_mp_{m+1}} \subset \{|\operatorname{Im}(z)| \geq 1\}$ but $p \notin \{|\operatorname{Im}(z)| \geq 1\}$. Therefore the set is not connected. ■

Solution. (c) Since $-2 \in \{1 \leq |z+3| < 2\}$ and $\{|z+2| < r\} \not\subset \{1 \leq |z+3| < 2\}$ for all $r > 0$, $\{1 \leq |z+3| < 2\}$ is not open. Similarly, -1 is a point lying on its complement

$$\{1 \leq |z+3| < 2\}^c = \{|z+3| \geq 2\} \cup \{|z+3| < 1\}$$

and $\{|z+1| < r\} \not\subset \{1 \leq |z+3| < 2\}^c$ for all $r > 0$. Hence $\{1 \leq |z+3| < 2\}^c$ is not open and $\{1 \leq |z+3| < 2\}$ is not closed. In summary, $\{1 \leq |z+3| < 2\}$ is neither open nor closed.
Since

$$|z| = |z+3-3| \leq |z+3| + |-3| < 5$$

for all $|z+3| < 2$, $\{1 \leq |z+3| < 2\} \subset \{|z| < 5\}$ and hence it is bounded.
The set is connected. To see this, we let $p_1 = -3/2, p_2 = -3+3i/2, p_3 = -9/2$ and $p_4 = -3-3i/2$. All these points lie on the circle $\{|z+3| = 3/2\}$ and hence lie in $\{1 \leq |z+3| < 2\}$.
It is easy to check that for every point $p \in \{1 \leq |z+3| < 2\}$, $\overline{pp_k} \subset \{1 \leq |z+3| < 2\}$ for at least one $p_k \in \{p_1, p_2, p_3, p_4\}$. So the set is connected. ■

6. Show that

$$|\sin z|^2 = (\sin x)^2 + (\sinh y)^2$$

for all complex numbers $z = x + yi$.

Proof.

$$\begin{aligned}
|\sin(z)|^2 &= |\sin(x+yi)|^2 = |\sin(x)\cos(yi) + \cos(x)\sin(yi)|^2 \\
&= |\sin(x)\cosh(y) - i\cos(x)\sinh(y)|^2 \\
&= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x(1+\sinh^2 y) + \cos^2 x \sinh^2 y \\
&= \sin^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y = (\sin x)^2 + (\sinh y)^2.
\end{aligned}$$

■

7. Show that

$$|\cos(z)|^2 = (\cos x)^2 + (\sinh y)^2$$

for all $z \in \mathbb{C}$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

Proof.

$$\begin{aligned}
|\cos(z)|^2 = |\cos(x+yi)|^2 &= |\cos(x)\cos(yi) - \sin(x)\sin(yi)|^2 \\
&= |\cos(x)\cosh(y) - i\sin(x)\sinh(y)|^2 \\
&= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
&= \cos^2 x(1+\sinh^2 y) + \sin^2 x \sinh^2 y \\
&= \cos^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y = (\cos x)^2 + (\sinh y)^2
\end{aligned}$$

■

8. Show that

$$\tan(z_1+z_2) = \frac{\tan z_1 + \tan z_2}{1-(\tan z_1)(\tan z_2)}$$

for all complex numbers z_1 and z_2 satisfying $z_1, z_2, z_1+z_2 \neq n\pi + \pi/2$ for any integer n.

Proof. Since

$$\begin{aligned}
\tan z_1 + \tan z_2 &= \frac{i(e^{-iz_1} - e^{iz_1})}{e^{iz_1} + e^{-iz_1}} + \frac{i(e^{-iz_2} - e^{iz_2})}{e^{iz_2} + e^{-iz_2}} \\
&= i\frac{(e^{-iz_1} - e^{iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{-iz_2} - e^{iz_2})(e^{iz_1} + e^{-iz_1})}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})} \\
&= -2i\frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})}
\end{aligned}$$

and

$$\begin{aligned}
1-(\tan z_1)(\tan z_2) &= 1 - \left(\frac{i(e^{-iz_1} - e^{iz_1})}{e^{iz_1} + e^{-iz_1}}\right)\left(\frac{i(e^{-iz_2} - e^{iz_2})}{e^{iz_2} + e^{-iz_2}}\right) \\
&= \frac{(e^{-iz_1} + e^{iz_1})(e^{-iz_2} + e^{iz_2}) + (e^{-iz_1} - e^{-iz_1})(e^{-iz_2} - e^{iz_2})}{(e^{-iz_1} + e^{iz_1})(e^{-iz_2} + e^{iz_2})} \\
&= 2\frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})},
\end{aligned}$$

we have

$$\frac{\tan z_1 + \tan z_2}{1-(\tan z_1)(\tan z_2)} = -i\frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}} = \tan(z_1+z_2).$$

Alternatively, we can argue as follows if we assume that the identity holds for z_1 and z_2 real. Let

$$F(z_1,z_2) = \tan(z_1+z_2) - \frac{\tan z_1 + \tan z_2}{1-(\tan z_1)(\tan z_2)}.$$

We assume that $F(z_1,z_2) = 0$ for all $z_1, z_2 \in \mathbb{R}$ with $z_1, z_2, z_1+z_2 \neq n\pi + \pi/2$.
Fixing $z_1 \in \mathbb{R}$, we let $f(z) = F(z_1,z)$. Then $f(z)$ is analytic in its domain

$$\mathbb{C}\backslash(\{n\pi+\pi/2\} \cup \{n\pi+\pi/2-z_1\}).$$

And we know that $f(z) = 0$ for z real. Therefore, by the uniqueness of analytic functions, $f(z) \equiv 0$ in its domain. So $F(z_1,z_2) = 0$ for all $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{C}$ in its domain.
Fixing $z_2 \in \mathbb{C}$, we let $g(z) = F(z,z_2)$. Then $g(z)$ is analytic in its domain

$$\mathbb{C}\backslash(\{n\pi+\pi/2\} \cup \{n\pi+\pi/2-z_2\}).$$

And we have proved that $g(z) = 0$ for z real. Therefore, by the uniqueness of analytic functions, $g(z) \equiv 0$ in its domain. Hence $F(z_1,z_2) = 0$ for all $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$ in its domain. ■

9. Find all the complex roots of the equation $\cos z = 3$.

   Solution. Since $\cos z = (e^{iz} + e^{-iz})/2$, it comes down to solve the equation $e^{iz} + e^{-iz} = 6$, i.e.,

   $$w + w^{-1} = 6 \Leftrightarrow w^2 - 6w + 1 = 0$$

   if we let $w = e^{iz}$. The roots of $w^2 - 6w + 1 = 0$ are $w = 3 \pm 2\sqrt{2}$. Therefore, the solutions for $\cos z = 3$ are

   $$iz = \log(3 \pm 2\sqrt{2}) \Leftrightarrow z = -i(\ln(3 \pm 2\sqrt{2}) + 2n\pi i) = 2n\pi - i\ln(3 \pm 2\sqrt{2})$$

   for n integers. ■

10. Calculate $\sin\left(\frac{\pi}{4} + i\right)$.

    Solution.

    $$\begin{aligned}
    \sin\left(\frac{\pi}{4} + i\right) &= \frac{1}{2i}(e^{i(\pi/4+i)} - e^{-i(\pi/4+i)}) \\
    &= \frac{1}{2i}(e^{-1}e^{\pi i/4} - ee^{-\pi i/4}) \\
    &= \frac{1}{2i}\left(e^{-1}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) - e(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})\right) \\
    &= \frac{\sqrt{2}}{4}\left(e + \frac{1}{e}\right) + \frac{\sqrt{2}}{4}\left(e - \frac{1}{e}\right)i
    \end{aligned}$$

    ■

11. Compute $\cos\left(\frac{\pi}{3}+i\right)$.

Solution.

$$\begin{aligned}\cos\left(\frac{\pi}{3}+i\right) &= \frac{1}{2}(e^{i(\pi/3+i)}+e^{-i(\pi/3+i)})\\ &= \frac{1}{2}(e^{-1}e^{\pi i/3}+ee^{-\pi i/3})\\ &= \frac{1}{2}\left(e^{-1}(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})+e(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})\right)\\ &= \frac{1}{4}\left(e+\frac{1}{e}\right)-\frac{\sqrt{3}i}{4}\left(e-\frac{1}{e}\right)\end{aligned}$$

■

12. Find i^i and its principal value.

Solution. We have

$$i^i = e^{i\log i} = e^{i(2n\pi i+\pi i/2)} = e^{-2n\pi-\pi/2}$$

for n integers and its principal value given by

$$i^i = e^{i\,\mathrm{Log}\,i} = e^{i(\pi i/2)} = e^{-\pi/2}.$$

■

13. Let $f(z)$ be the principal branch of $\sqrt[3]{z}$.

(a) Find $f(-i)$.

Solution.

$$f(-i) = \exp(\frac{1}{3}\mathrm{Log}(-i)) = \exp(\frac{1}{3}(-\frac{\pi i}{2})) = \exp(-\frac{\pi i}{6}) = \frac{\sqrt{3}}{2}-\frac{i}{2}$$

■

(b) Show that

$$f(z_1)f(z_2) = \lambda f(z_1z_2)$$

for all $z_1, z_2 \neq 0$, where $\lambda = 1, \frac{-1+\sqrt{3}i}{2}$ or $\frac{-1-\sqrt{3}i}{2}$.

Proof. Since

$$\begin{aligned}\frac{f(z_1)f(z_2)}{f(z_1z_2)} &= \exp(\frac{1}{3}\operatorname{Log} z_1 + \frac{1}{3}\operatorname{Log} z_2 - \frac{1}{3}\operatorname{Log}(z_1z_2)) \\ &= \exp(\frac{1}{3}(\operatorname{Log} z_1 + \operatorname{Log} z_2 - \operatorname{Log}(z_1z_2))) \\ &= \exp(\frac{i}{3}(\operatorname{Arg} z_1 + \operatorname{Arg} z_2 - \operatorname{Arg}(z_1z_2))) = \exp(\frac{2n\pi i}{3})\end{aligned}$$

for some integer n, $\lambda = \exp(2n\pi i/3)$. Therefore,

$$\lambda = \begin{cases} 1 & \text{if } n = 3k \\ \frac{-1+\sqrt{3}i}{2} & \text{if } n = 3k+1 \\ \frac{-1-\sqrt{3}i}{2} & \text{if } n = 3k+2 \end{cases}$$

where $k \in \mathbb{Z}$. ■

14. Let $f(z)$ be the principal branch of z^{-i}.

(a) Find $f(i)$.

Solution.

$$f(i) = i^{-i} = \exp(-i\operatorname{Log}(i)) = \exp(-i(\pi i/2)) = e^{\pi/2}.$$

■

(b) Show that

$$f(z_1)f(z_2) = \lambda f(z_1z_2)$$

for all $z_1, z_2 \neq 0$, where $\lambda = 1, e^{2\pi}$ or $e^{-2\pi}$.

Proof. Since

$$\begin{aligned}\frac{f(z_1)f(z_2)}{f(z_1z_2)} &= \exp(-i\operatorname{Log} z_1 - i\operatorname{Log} z_2 + i\operatorname{Log}(z_1z_2)) \\ &= \exp(-i(\operatorname{Log} z_1 + \operatorname{Log} z_2 - \operatorname{Log}(z_1z_2))) \\ &= \exp(-i(i\operatorname{Arg} z_1 + i\operatorname{Arg} z_2 - i\operatorname{Arg}(z_1z_2))) \\ &= \exp(\operatorname{Arg} z_1 + \operatorname{Arg} z_2 - \operatorname{Arg}(z_1z_2)) \\ &= \exp(2n\pi)\end{aligned}$$

for some integer n, $\lambda = \exp(2n\pi)$. And since

$$-\pi < \operatorname{Arg}(z_1) \leq \pi, -\pi < \operatorname{Arg}(z_2) \leq \pi$$

and

$$-\pi < \operatorname{Arg}(z_1z_2) \leq \pi,$$

we conclude that

$$-3\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - \operatorname{Arg}(z_1z_2) < 3\pi$$

and hence $-3 < 2n < 3$. So $n = -1, 0$ or 1 and $\lambda = e^{-2\pi}, 1$ or $e^{2\pi}$. ■

15. Determine the Möbius transformation mapping 0 to 2, $2i$ to 0, and i to 3/2.

Solution. Consider the function

$$f(z) = \frac{az+b}{cz+d}. \tag{2.1}$$

Then

$$\begin{aligned} f(0) = 2 \quad &\text{then} \quad \frac{b}{d} = 2 && (2.2)\\ f(-2) = 0 \quad &\text{then} \quad -2ia + b = 0 && (2.3)\\ f(i) = \frac{3}{2} \quad &\text{then} \quad \frac{ai+b}{ci+d} = \frac{3}{2}. && (2.4) \end{aligned}$$

From 2.2 we get $d \neq 0$. Notice also that we can take d to be any nonzero complex number.
Let us take $d = i$. Thus, from expression 2.2, we have $b = 2i$. From 2.3, $a = 1$, and from 2.4 we have $c = 1$.
Hence the transformation we are looking for is

$$f(z) = \frac{z+2i}{z+i}.$$

Remark: Other formulations are possible for different choices of the constant d, e.g.

$$f(z) = \frac{iz-2}{iz-1}.$$

■

16. Let T be a mapping from $\mathbb{C}$ to $\mathbb{C}$. A fixed point of T is a point z satisfying $T(z) = z$.
   (a) Show that any Möbius transformation, apart from the identity, can have at most 2 fixed points in $\mathbb{C}$.
   (b) Give examples of Möbius transformations having (*i*) 2; (*ii*) 1 and (*iii*) no fixed points in $\mathbb{C}$.

Proof. (a) Let the Möbius transformation be

$$T(z) = \frac{az+b}{cz+d}.$$

If $T(z) = z$, then

$$\begin{aligned} \frac{az+b}{cz+d} &= z,\\ cz^2 + dz &= az + b,\\ cz^2 + (d-a)z - b &= 0. \end{aligned} \tag{2.5}$$

Now, if $c \neq 0$, then the expression 2.5 is a quadratic equation with one or two solutions. That is

$$\frac{-(d-a) + [(d-a)^2 + 4bc]^{1/2}}{2c}.$$

If $c = 0$, expression 2.5 becomes

$$(d-a)z = b.$$

This equation has one solution $b/(d-a)$, if $d \neq a$. It has no solution, if $d = a$ and $b \neq 0$. And, finally, it has infinitely many solutions, if $d = a$ and $b = 0$.
Notice that in the last case we have $d = a$ and $b = c = 0$. Since $ad - bc \neq 0$, then $a = d \neq 0$. Thus we have

$$T(z) = \frac{az}{d} = z.$$

That is the identity, which is excluded.
(b) (i) From part (a), if $c \neq 0$ and $(d-a)^2 + 4bc \neq 0$, we will have 2 fixed points. Hence, the function

$$T(z) = \frac{1}{z}$$

is an example with two fixed points: -1 and 1.
(ii) Now, a function with one fixed point, is the following

$$T(z) = \frac{z}{2}.$$

(iii) Finally, with no fixed points, we have

$$T(z) = z + 1.$$

■

17. For $z \in \mathbb{C}$, show that:
   (a) $\sin \overline{z} = \overline{\sin z}$;
   (b) $\cosh \overline{z} = \overline{\cosh z}$.

Proof. (a) Using the definition of $\sin z$, we have

$$\begin{aligned}
\sin \overline{z} &= \frac{e^{i\overline{z}} - e^{-i\overline{z}}}{2i} = \frac{e^{\overline{-iz}} - e^{\overline{iz}}}{\overline{-2i}} = \overline{\left(\frac{e^{-iz} - e^{iz}}{-2i}\right)} = \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} \\
&= \overline{\sin z}
\end{aligned}$$

(b) Recall that for $z = x + iy$, we have that

$$e^{\overline{z}} = e^{x-iy} = e^x e^{-iy} = \overline{e^x e^{iy}} = \overline{e^z}.$$

Thus

$$\cosh \overline{z} = \frac{e^{\overline{z}} - e^{-\overline{z}}}{2} = \overline{\left(\frac{e^z - e^{-z}}{2}\right)} = \overline{\cosh z}$$

■

18. Find all solutions $z \in \mathbb{C}$ of of the following (express your answers in the form $x+iy$):
   (a) $\log z = 4i$;
   (b) $z^i = i$.

Solution. (a) We have that $\exp(\log z) = z$. Thus

$$z = \exp(\log z) = \exp(4i) = \cos 4 + i \sin 4$$

Notice that, if $z = \exp(4i)$, then we have

$$\log(\exp(4i)) = 4i + 2n\pi i \qquad (n \in \mathbb{Z})$$

In particular, for $n = 0$, we have that $\log[\exp(4i)] = 4i$.

(b) **Method one:** We know that $z^i = \exp(i \log z)$. Thus

$$\exp(i \log z) \quad = \quad i$$

Since $i = \exp(i(\pi/2 + 2n\pi))$, with $n \in \mathbb{Z}$, then we have

$$\begin{aligned}\exp\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right) &= \exp[i(\ln|z| + i\arg(z))] \\ &= \exp[-\arg(z) + i\ln|z|] \\ &= \exp[-\arg(z)] \cdot \exp[i\ln|z|]\end{aligned}$$

Thus

$$\arg(z) = 2k\pi \;\; (k \in \mathbb{Z}) \quad \text{and} \quad \ln|z| = \frac{\pi}{2} + 2n\pi \;\; (n \in \mathbb{Z})$$

Hence,

$$z = \exp\left[\frac{\pi}{2} + 2n\pi\right] \qquad (n \in \mathbb{Z}).$$

Method two: Consider the following identity

$$\log[\exp(i \log z)] \quad = \quad \log i \tag{2.6}$$

Thus

$$\log[\exp(i \log z)] = i \log z + 2n_1\pi i \qquad (n_1 \in \mathbb{Z})$$

Hence, substituting in (2.6), we obtain

$$i \log z + 2n_1\pi i \quad = \quad \log i \qquad (n_1 \in \mathbb{Z})$$

Thus

$$\log z = -i \log i - 2n_1\pi \qquad (n_1 \in \mathbb{Z})$$

From the polar form of i, we have that $r = 1$ and $\Theta = \frac{\pi}{2}$. Thus

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n_2\pi\right) \qquad (n_2 \in \mathbb{Z})$$

So

$$\log z = -i\left[\ln 1 + i\left(\frac{\pi}{2} + 2n_2\pi\right)\right] - 2n_1\pi = \frac{\pi}{2} + 2\pi(n_2 - n_1)$$

with $n_1, n_2 \in \mathbb{Z}$. Since $n_2 - n_1 \in \mathbb{Z}$, we have

$$\log z = \frac{\pi}{2} + 2n\pi \qquad (n \in \mathbb{Z})$$

Therefore, $z^i = i$ when

$$z = \exp\left[\frac{\pi}{2} + 2n\pi\right] \qquad (n \in \mathbb{Z})$$

■

19. Show that

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right). \tag{2.7}$$

Solution. If $w = \tanh^{-1} z$, then

$$z = \tanh w = \frac{\sinh w}{\cosh w} = \frac{e^w - e^{-w}}{e^w + e^{-w}} = \frac{e^{2w}-1}{e^{2w}+1}$$

Thus

$$\begin{aligned} z(e^{2w}+1) &= e^{2w}-1 \\ e^{2w} &= \frac{z+1}{1-z} \\ 2w\log e &= \log\left(\frac{z+1}{1-z}\right) \\ w &= \frac{1}{2}\log\left(\frac{z+1}{1-z}\right) \end{aligned}$$

Hence

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{1-z}\right).$$

■

20. Find all solutions of the equation $\tanh z = i$ and express them in the form $x + iy$.

Solution. **Method one:**

Applying the inverse hyperbolic function in both sides we have

$$z = \tanh^{-1} z = \tanh^{-1}(i).$$

Using the formula (2.7), we have

$$\begin{aligned} \tanh^{-1}(i) &= \frac{1}{2}\log\left(\frac{1+i}{1-i}\right) \\ &= \frac{1}{2}\log(i) \end{aligned}$$

Using the definition of log for i with $r = 1$ and $\Theta = \pi/2$ we have that

$$\log(i) = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(\frac{\pi}{2} + 2n\pi\right) \quad (n \in \mathbb{Z}).$$

Therefore

$$z = \tanh^{-1}(i) = \frac{1}{2}\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right] = i\left(\frac{\pi}{4} + n\pi\right)$$

for $n \in \mathbb{Z}$.

Method two:

We know that

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{-i\sin(iz)}{\cos(iz)} = -i\tan(iz).$$

Thus,

$$\tanh z = i \quad \Longleftrightarrow \quad -i\tan(iz) = i \quad \Longleftrightarrow \quad \tan(iz) = -1.$$

Applying the inverse trigonometric function in both sides we have

$$iz = \tan^{-1}(iz) = \tan^{-1}(-1). \tag{2.8}$$

Now, using the formula

$$\tan^{-1} z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

we obtain

$$\begin{aligned}
\tan^{-1}(-1) &= \frac{i}{2}\log\left(\frac{i-1}{1+i}\right) \\
&= \frac{i}{2}\log(i) \\
&= \frac{i}{2}\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right] \quad (n \in \mathbb{Z}) \\
&= -\left(\frac{\pi}{4} + n\pi\right) \quad (n \in \mathbb{Z}).
\end{aligned}$$

Thus, substituting in (2.8), we have

$$iz = -\left(\frac{\pi}{4} + n\pi\right) \quad (n \in \mathbb{Z}).$$

Hence

$$z = i\left(\frac{\pi}{4} + n\pi\right) \quad (n \in \mathbb{Z}).$$

■

21. Let Ω_1 and Ω_2 be nonempty, closed sets in $\mathbb{C}$.
    (a) Show that the set $\Omega_1 \cup \Omega_2$ is closed.
    (b) If instead Ω_2 is nonempty and open:
        (i) could $\Omega_1 \cup \Omega_2$ still be closed?
        (ii) Need it be closed?
        Give proofs or examples/counterexamples.

Proof. (a) We need to prove that

$$\partial(\Omega_1 \cup \Omega_2) \subseteq \Omega_1 \cup \Omega_2.$$

Let $z \in \partial(\Omega_1 \cup \Omega_2)$. Thus, for every $\varepsilon > 0$, we have that

$$B_\varepsilon(z) \cap (\Omega_1 \cup \Omega_2) \neq \emptyset \tag{2.9}$$

and

$$B_\varepsilon(z) \cap (\Omega_1 \cup \Omega_2)^c = B_\varepsilon(z) \cap (\Omega_1^c \cap \Omega_2^c) \neq \emptyset. \tag{2.10}$$

From expression (2.9),

$$B_\varepsilon(z) \cap \Omega_1 \neq \emptyset \quad \text{or} \quad B_\varepsilon(z) \cap \Omega_2 \neq \emptyset, \quad \text{or both.}$$

But, from expression (2.10), we have

$$B_\varepsilon(z) \cap \Omega_1^c \neq \emptyset \quad \text{and} \quad B_\varepsilon(z) \cap \Omega_2^c \neq \emptyset.$$

Hence

$$B_\varepsilon(z) \cap \Omega_1 \neq \emptyset \quad \text{and} \quad B_\varepsilon(z) \cap \Omega_1^c \neq \emptyset,$$

or

$$B_\varepsilon(z) \cap \Omega_2 \neq \emptyset \quad \text{and} \quad B_\varepsilon(z) \cap \Omega_2^c \neq \emptyset,$$

or both. This means that $z \in \partial\Omega_1$ or $z \in \partial\Omega_2$ (or both). That is

$$z \in \partial\Omega_1 \cup \partial\Omega_2.$$

Since Ω_1 and Ω_2 are closed, we have that $z \in \partial\Omega_1 \cup \partial\Omega_2 \subseteq \Omega_1 \cup \Omega_2$. Therefore $\Omega_1 \cup \Omega_2$ is closed.
(b) (i) The answer is 'yes'. For example, $\Omega_1 = \overline{B}_1(0)$, $\Omega_2 = B_1(0)$.
(ii) The answer is 'no'. For example, $\Omega_1 = \overline{B}_1(0)$, $\Omega_2 = B_2(0)$. In this case, $\Omega_1 \cup \Omega_2 = \Omega_2$ is open. ■

2.2 Limits, Continuity and Differentiation

1. Compute the following limits if they exist:

   (a) $\lim_{z\to -i} \dfrac{iz^3+1}{z^2+1}$;

   (b) $\lim_{z\to\infty} \dfrac{4+z^2}{(z-1)^2}$.

   (c) $\lim_{z\to 0} \dfrac{\mathrm{Im}(z)}{z}$.

   Solution. (a)

$$\begin{aligned}
\lim_{z\to -i} \frac{iz^3+1}{z^2+1} &= \lim_{z\to -i} \frac{i(z^3+i^3)}{z^2+1} \\
&= \lim_{z\to -i} \frac{i(z+i)(z^2-iz+i^2)}{(z+i)(z-i)} \\
&= \lim_{z\to -i} \frac{i(z^2-iz+i^2)}{z-i} \\
&= i\frac{\lim_{z\to -i}(z^2-iz+i^2)}{\lim_{z\to -i}(z-i)} \\
&= i\frac{\lim_{z\to -i} z^2 - i\lim_{z\to -i} z + \lim_{z\to -i} i^2}{\lim_{z\to -i} z - \lim_{z\to -i} i} \\
&= \frac{i((-i)^2-i(-i)+i^2)}{-i-i} = \frac{3}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{z\to\infty} \frac{4+z^2}{(z-1)^2} &= \lim_{z\to 0} \frac{4+z^{-2}}{(z^{-1}-1)^2} \\
&= \lim_{z\to 0} \frac{4z^2+1}{(1-z)^2} = \frac{\lim_{z\to 0}(4z^2+1)}{\lim_{z\to 0}(1-z)^2} \\
&= \frac{4(\lim_{z\to 0} z)^2 + \lim_{z\to 0} 1}{(\lim_{z\to 0} 1 - \lim_{z\to 0} z)^2} = 1
\end{aligned}$$

(c) Since

$$\lim_{\mathrm{Re}(z)=0, z\to 0} \frac{\mathrm{Im}(z)}{z} = \lim_{y\to 0} \frac{y}{yi} = -i$$

and

$$\lim_{\mathrm{Im}(z)=0, z\to 0} \frac{\mathrm{Im}(z)}{z} = \lim_{x\to 0} \frac{0}{x} = 0$$

the limit does not exist.

2. Show the following limits:
   (a) $\lim_{z\to\infty} \frac{4z^5}{z^5 - 42z} = 4$;
   (b) $\lim_{z\to\infty} \frac{z^4}{z^2 + 42z} = \infty$;
   (c) $\lim_{z\to\infty} \frac{(az+b)^3}{(cz+d)^3} = \frac{a^3}{c^3}$, if $c \neq 0$.

   Solution. (a)

$$\lim_{z\to\infty} \frac{4z^5}{z^5 - 42z} = \lim_{z\to 0} \frac{4\left(\frac{1}{z}\right)^5}{\left(\frac{1}{z}\right)^5 - 42\left(\frac{1}{z}\right)} = \lim_{z\to 0} \frac{4}{1 - 42z^4} = 4.$$

   (b) Notice that

$$\begin{aligned} \lim_{z\to\infty} \frac{z^4}{z^2+42z} \quad &\iff \quad \lim_{z\to 0} \left[\frac{1/z^4}{1/z^2 + 42/z}\right]^{-1} \\ &\iff \quad \lim_{z\to 0} \left[\frac{1}{z^2 + 42z^3}\right]^{-1} \\ &\iff \quad \lim_{z\to 0} \left(z^2 + 42z^3\right) = 0. \end{aligned}$$

   Therefore, $\lim_{z\to\infty} \frac{z^4}{z^2+42z} = \infty$.
   (c)

$$\lim_{z\to\infty} \frac{(az+b)^3}{(cz+d)^3} = \lim_{z\to 0} \frac{(a/z+b)^3}{(c/z+d)^3} = \lim_{z\to 0} \frac{(a+bz)^3}{(c+dz)^3} = \frac{a^3}{c^3}.$$

■

3. Show that $\lim_{z\to 0} z/\bar{z}$ does not exist.
   Hint: Consider what happens to the function at points of the form $x + 0i$ for $x \to 0$, $x \neq 0$, and then at points of the form $0 + yi$ for $y \to 0$, $y \neq 0$.

   Proof. For $z = x + 0i$, $x \neq 0$,

$$\frac{z}{\bar{z}} = \frac{x}{x} = 1 \to 1 \quad \text{as} \quad x \to 0.$$

   On the other hand, for $z = 0 + yi$, $y \neq 0$,

$$\frac{z}{\bar{z}} = \frac{yi}{-yi} = -1 \to -1 \quad \text{as} \quad y \to 0.$$

   However, $\lim_{z\to 0} z/\bar{z}$ must be independent of direction of approach. Hence limit does not exist. ■

4. Show that if $f(z)$ is continuous at z_0, so is $|f(z)|$.

Proof. Let $f(z) = u(x,y) + iv(x,y)$. Since $f(z)$ is continuous at $z_0 = x_0 + y_0 i$, $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0). Therefore,

$$(u(x,y))^2 + (v(x,y))^2$$

is continuous at (x_0, y_0) since the sums and products of continuous functions are continuous. It follows that

$$|f(z)| = \sqrt{(u(x,y))^2 + (v(x,y))^2}$$

is continuous at z_0 since the compositions of continuous functions are continuous. ■

5. Let

$$f(z) = \begin{cases} \overline{z}^3/z^2 & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that
(a) $f(z)$ is continuous everywhere on $\mathbb{C}$;
(b) the complex derivative $f'(0)$ does not exist.

Proof. Since both $\overline{z}^3$ and z^2 are continuous on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $z^2 \neq 0$, $f(z) = \overline{z}^3/z^2$ is continuous on $\mathbb{C}^*$.
At $z = 0$, we have

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \left| \frac{\overline{z}^3}{z^2} \right| = \lim_{z\to 0} |z| = 0$$

and hence $\lim_{z\to 0} f(z) = 0 = f(0)$. So f is also continuous at 0 and hence continuous everywhere on $\mathbb{C}$.
The complex derivative $f'(0)$, if exists, is given by

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\overline{z}^3}{z^3}.$$

Let $z = x + yi$. If $y = 0$ and $x \to 0$, then

$$\lim_{z=x\to 0} \frac{\overline{z}^3}{z^3} = \lim_{x\to 0} \frac{x^3}{x^3} = 1.$$

On the other hand, if $x = 0$ and $y \to 0$, then

$$\lim_{z=yi\to 0} \frac{\overline{z}^3}{z^3} = \lim_{x\to 0} \frac{(-yi)^3}{(yi)^3} = -1.$$

So the limit $\lim_{z\to 0} \overline{z}^3/z^3$ and hence $f'(0)$ do not exist. ■

6. Show that $f(z)$ in (2) is actually nowhere differentiable, i.e., the complex derivative $f'(z)$ does not exist for any $z \in \mathbb{C}$.

   Proof. It suffices to show that C-R equations fail at every $z \neq 0$:

$$
\begin{aligned}
\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z) &= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \frac{\bar{z}^3}{z^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\bar{z}^3}{z^2}\right) + i\frac{\partial}{\partial y}\left(\frac{\bar{z}^3}{z^2}\right) \\
&= \left(\frac{3\bar{z}^2}{z^2} - \frac{2\bar{z}^3}{z^3}\right) + i\left(-\frac{3i\bar{z}^2}{z^2} - \frac{2i\bar{z}^3}{z^3}\right) \\
&= \frac{6\bar{z}^2}{z^2} \neq 0
\end{aligned}
$$

   for $z \neq 0$. ■

7. Find $f'(z)$ when
   (a) $f(z) = z^2 - 4z + 2$;
   (b) $f(z) = (1 - z^2)^4$;
   (c) $f(z) = \dfrac{z+1}{2z+1}$ $(z \neq -\dfrac{1}{2})$;
   (d) $f(z) = e^{1/z}$ $(z \neq 0)$.

   Answer. (a) $2z - 4$; (b) $-8(1-z^2)^3 z$; (c) $-1/(2z+1)^2$; (d) $-e^{1/z}/z^2$.

8. Prove the following version of complex L'Hospital: Let $f(z)$ and $g(z)$ be two complex functions defined on $|z - z_0| < r$ for some $r > 0$. Suppose that $f(z_0) = g(z_0) = 0$, $f(z)$ and $g(z)$ are differentiable at z_0 and $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

   [Refer to: problems 1c and 6 in section 3.1; and problem 12 in section 3.2]

   Proof. Since $f(z)$ and $g(z)$ are differentiable at z_0, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

   and

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = g'(z_0).$$

   And since $g'(z_0) \neq 0$,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \frac{\lim_{z \to z_0}(f(z) - f(z_0))/(z - z_0)}{\lim_{z \to z_0}(g(z) - g(z_0))/(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}.$$

   Finally, since $f(z_0) = g(z_0) = 0$,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

   ■

9. Show that if $f(z)$ satisfies the Cauchy-Riemann equations at z_0, so does $(f(z))^n$ for every positive integer n.

Proof. Since $f(z)$ satisfies the Cauchy-Riemann equations at z_0,

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z)=0$$

at z_0. Therefore,

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(f(z))^n &= \frac{\partial}{\partial x}(f(z))^n + i\frac{\partial}{\partial y}(f(z))^n\\ &= (f(z))^{n-1}\frac{\partial}{\partial x}f(z)+i(f(z))^{n-1}\frac{\partial}{\partial y}f(z)\\ &= (f(z))^{n-1}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f(z)=0\end{aligned}$$

at z_0. ■

2.3 Analytic functions

1. Explain why the function $f(z)=2z^2-3-ze^z+e^{-z}$ is entire.

Proof. Since every polynomial is entire, $2z^2-3$ is entire; since both $-z$ and e^z are entire, their product $-ze^z$ is entire; since e^z and $-z$ are entire, their composition e^{-z} is entire. Finally, $f(z)$ is the sum of $2z^3-3$, $-ze^z$ and e^{-z} and hence entire. ■

2. Let $f(z)$ be an analytic function on a connected open set D. If there are two constants c_1 and $c_2\in\mathbb{C}$, not all zero, such that $c_1f(z)+c_2\overline{f(z)}=0$ for all $z\in D$, then $f(z)$ is a constant on D.

Proof. If $c_2=0$, $c_1\neq 0$ since c_1 and c_2 cannot be both zero. Then we have $c_1f(z)=0$ and hence $f(z)=0$ for all $z\in D$.
If $c_2\neq 0$, $\overline{f(z)}=-(c_1/c_2)f(z)$. And since $f(z)$ is analytic in D, $\overline{f(z)}$ is anlaytic in D. So both $f(z)$ and $\overline{f(z)}$ are analytic in D. Therefore, both $f(z)$ and $\overline{f(z)}$ satisfy Cauchy-Riemann equations in D. Hence

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u+vi)=0$$

and

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u-vi)=0$$

in D, where $f(z)=u(x,y)+iv(x,y)$ with $u=\mathrm{Re}(f)$ and $v=\mathrm{Im}(f)$. It follows that

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)u=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)v=0$$

and hence $u_x=u_y=v_x=v_y=0$ in D. Therefore, u and v are constants on D and hence $f(z)\equiv\text{const}$. ■

3. Show that the function $\sin(\bar{z})$ is nowhere analytic on $\mathbb{C}$.

   Proof. Since

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\sin(\bar{z}) &= \frac{\partial}{\partial x}\sin(\bar{z})+i\frac{\partial}{\partial y}\sin(\bar{z})\\ &= \cos(\bar{z})\frac{\partial \bar{z}}{\partial x}+i\cos(\bar{z})\frac{\partial \bar{z}}{\partial y}\\ &= \cos(\bar{z})+i\cos(\bar{z})(-i)=2\cos(\bar{z})\end{aligned}$$

   $\sin(\bar{z})$ is not differentiable and hence not analytic at every point z satisfying $\cos(\bar{z})\neq 0$. At every point z_0 satisfying $\cos(\bar{z}_0)=0$, i.e., $z_0=n\pi+\pi/2$, $\sin(\bar{z})$ is not differentiable in $|z-z_0|<r$ for all $r>0$. Hence $\sin(\bar{z})$ is not analytic at $z_0=n\pi+\pi/2$ either. In conclusion, $\sin(\bar{z})$ is nowhere analytic. ■

4. Let $f(z)=u(x,y)+iv(x,y)$ be an entire function satisfying $u(x,y)\le x$ for all $z=x+yi$. Show that $f(z)$ is a polynomial of degree at most one.

   Proof. Let $g(z)=\exp(f(z)-z)$. Then $|g(z)|=\exp(u(x,y)-x)$. Since $u(x,y)\le x$, $|g(z)|\le 1$ for all z. And since $g(z)$ is entire, $g(z)$ must be constant by Louville's theorem. Therefore, $g'(z)\equiv 0$. That is, $(f'(z)-1)\exp(f(z)-z)\equiv 0$ and hence $f'(z)=1$ for all z. So $f(z)\equiv z+c$ for some constant c. ■

5. Show that

$$|\exp(z^3+i)+\exp(-iz^2)|\le e^{x^3-3xy^2}+e^{2xy}$$

   where $x=\mathrm{Re}(z)$ and $y=\mathrm{Im}(z)$.

   Proof. Note that $|e^z|=e^{\mathrm{Re}(z)}$. Therefore,

$$\begin{aligned}|\exp(z^3+i)+\exp(-iz^2)| &\le |\exp(z^3+i)|+|\exp(-iz^2)|\\ &= \exp(\mathrm{Re}(z^3+i))+\exp(\mathrm{Re}(-iz^2))\\ &= \exp(\mathrm{Re}((x^3-3xy^2)+(3x^2y-y^3+1)i))\\ &\quad+\exp(\mathrm{Re}(2xy-(x^2-y^2)i))\\ &= e^{x^3-3xy^2}+e^{2xy}.\end{aligned}$$

   ■

6. Let $f(z)=u(x,y)+iv(x,y)$ be an entire function satisfying

$$v(x,y)\ge x$$

   for all $z=x+yi$, where $u(x,y)=\mathrm{Re}(f(z))$ and $v(x,y)=\mathrm{Im}(f(z))$. Show that $f(z)$ is a polynomial of degree 1.

Proof. Let $g(z) = \exp(if(z)+z)$. Then

$$|g(z)| = |\exp((-v(x,y)+iu(x,y))+(x+iy))| = \exp(x-v(x,y)).$$

Since $v(x,y) \le x$, $x-v(x,y) \le 0$ and $|g(z)| \le 1$ for all z. And since $g(z)$ is entire, $g(z)$ must be constant by Louville's theorem. Therefore, $g'(z) \equiv 0$. That is, $(if'(z)+1)\exp(if(z)+z) \equiv 0$ and hence $f'(z) = i$ for all z. So $f(z) \equiv iz+c$ for some constant c. ■

7. Show that the entire function $\cosh(z)$ takes every value in $\mathbb{C}$ infinitely many times.

   Proof. For every $w_0 \in \mathbb{C}$, the quadratic equation $y^2 - 2w_0 y + 1 = 0$ has a complex root y_0. We cannot have $y_0 = 0$ since $0^2 - 2w_0 \cdot 0 + 1 \neq 0$. Therefore, $y_0 \neq 0$ and there is $z_0 \in \mathbb{C}$ such that $e^{z_0} = y_0$. Then

   $$\cosh(z_0) = \frac{e^{z_0}+e^{-z_0}}{2} = \frac{y_0^2+1}{2y_0} = \frac{2w_0 y_0}{2y_0} = w_0.$$

   And since $\cosh(z+2\pi i) = \cosh(z)$, $\cosh(z_0+2n\pi i) = w_0$ for all integers n. Therefore, $\cosh(z)$ takes every value w_0 infinitely many times. ■

8. Determine which of the following functions $f(z)$ are entire and which are not? You must justify your answer. Also find the complex derivative $f'(z)$ of $f(z)$ if $f(z)$ is entire. Here $z = x+yi$ with $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

   (a) $f(z) = \dfrac{1}{1+|z|^2}$

   Solution. Since $u(x,y) = \mathrm{Re}(f(z)) = (1+x^2+y^2)^{-1}$ and $v(x,y) = 0$, $u_x = 2x(1+x^2+y^2)^{-2} \neq 0 = v_y$. Hence the Cauchy-Riemann equations fail for $f(z)$ and $f(z)$ is not entire.

   (b) $f(z) = 2^{(3^z)}$ (here 2^z and 3^z are taken to be the principle values of 2^z and 3^z, respectively, by convention)

   Solution. Let $g(z) = 2^z$ and $h(z) = 2^{3^z}$. Since both $g(z)$ and $h(z)$ are entire, $f(z) = g(h(z))$ is entire and

   $$f'(z) = g'(h(z))h'(z) = 2^{3^z}3^z(\ln 2)(\ln 3)$$

   by chain rule.

   (c) $f(z) = (x^2-y^2) - 2xyi$

   Solution. Since

   $$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = (2x-2yi)+i(-2y-2xi) = 4x-4yi \neq 0,$$

   the Cauchy-Riemann equations fail for $f(z)$ and hence $f(z)$ is not entire.

(d) $f(z) = (x^2 - y^2) + 2xyi$

Solution. Since

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = (2x - 2yi) + i(2y + 2xi) = 0,$$

the Cauchy-Riemann equations hold for $f(z)$ everywhere. And since f_x and f_y are continous, $f(z)$ is analytic on $\mathbb{C}$. And $f'(z) = f_x = 2x + 2yi = 2z$.

9. Let C_R denote the upper half of the circle $|z| = R$ for some $R > 1$. Show that

$$\left|\frac{e^{iz}}{z^2 + z + 1}\right| \le \frac{1}{(R-1)^2}$$

for all z lying on C_R.

Proof. For $z \in C_R$, $|z| = R$ and $\operatorname{Im}(z) \ge 0$. Let $z = x + yi$. Since $y = \operatorname{Im}(z) \ge 0$,

$$|e^{iz}| = |e^{i(x+yi)}| = |e^{-y+xi}| = e^{-y} \le 1$$

for $z \in C_R$. And since

$$\begin{aligned}
|z^2 + z + 1| &= \left|\left(z - \frac{-1+\sqrt{3}i}{2}\right)\left(z - \frac{-1-\sqrt{3}i}{2}\right)\right| \\
&= \left|z - \frac{-1+\sqrt{3}i}{2}\right|\left|z - \frac{-1-\sqrt{3}i}{2}\right| \\
&\ge \left(|z| - \left|\frac{-1+\sqrt{3}i}{2}\right|\right)\left(|z| - \left|\frac{-1-\sqrt{3}i}{2}\right|\right) \\
&= (R-1)(R-1) = (R-1)^2,
\end{aligned}$$

we obtain

$$\left|\frac{e^{iz}}{z^2 + z + 1}\right| \le \frac{1}{(R-1)^2}$$

for $z \in C_R$. ■

10. Let

$$f(z) = \begin{cases} \overline{z}^2/|z| & \text{if } z \ne 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that $f(z)$ is continuous everywhere but nowhere analytic on $\mathbb{C}$.

Proof. Since both $\bar{z}$ and $|z|$ are continuous on $\mathbb{C}$, $\bar{z}^2/|z|$ is continuous on $\mathbb{C}^*$. Therefore, $f(z)$ is continuous on $\mathbb{C}^*$. To see that it is continuous at 0, we just have to show that

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\bar{z}^2}{|z|} = f(0) = 0.$$

This follows from

$$\lim_{z\to 0} \left|\frac{\bar{z}^2}{|z|}\right| = \lim_{z\to 0} \frac{|\bar{z}|^2}{|z|} = \lim_{z\to 0} |z| = 0.$$

Therefore, $f(z)$ is continuous everywhere on $\mathbb{C}$.
To show that $f(z)$ is nowhere analytic, it suffices to show that the Cauchy-Riemann equations fail for $f(z)$ on $\mathbb{C}^*$. This follows from

$$\begin{aligned} &\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\bar{z}^2}{|z|}\right) \\ &= \left(\frac{2\bar{z}}{|z|} - \frac{x\bar{z}^2}{|z|^3}\right) + i\left(-\frac{2i\bar{z}}{|z|} - \frac{y\bar{z}^2}{|z|^3}\right) \\ &= \frac{(4z - x - iy)\bar{z}^2}{|z|^3} = \frac{3\bar{z}}{|z|} \neq 0 \end{aligned}$$

for $z \neq 0$. Consequently, $f(z)$ is nowhere analytic. ■

11. Find where

$$\tan^{-1}(z) = \frac{i}{2}\operatorname{Log}\frac{i+z}{i-z}$$

is analytic?

Solution. The branch locus of $\tan^{-1}(z)$ is

$$\left\{z : \frac{i+z}{i-z} = w \in (-\infty, 0]\right\} = \left\{z : z = i\frac{w-1}{w+1}, w \in (-\infty, 0]\right\}.$$

For $w \in (-\infty, 0]$,

$$\frac{w-1}{w+1} = 1 - \frac{2}{w+1} \in (-\infty, -1] \cup (1, \infty)$$

so $\tan^{-1}(z)$ is analytic in

$$\mathbb{C} \setminus \{z : \operatorname{Re}(z) = 0, \operatorname{Im}(z) \in (-\infty, -1] \cup [1, \infty)\}.$$

12. Show that the following functions are defined on all of $\mathbb{C}$, but are nowhere analytic (here $z = x + iy$):
   (a) $z \mapsto 2xy + i(x^2 + y^2)$;
   (b) $z \mapsto e^y e^{ix}$.

   Proof. (a) Notice that the real and imaginary components of the function

$$f(z) = f(x+iy) = 2xy + i(x^2 + y^2)$$

are well defined. Hence the function $f(z)$ is defined on all $\mathbb{C}$. If $f(z)$ were analytic then we will have

$$u_x = v_y \quad \Rightarrow \quad 2y = 2y \tag{2.11}$$

and

$$u_y = -v_x \quad \Rightarrow \quad 2x = -2x \tag{2.12}$$

Equation (2.11) is true for all $z \in \mathbb{C}$. However, equation (2.12) is true only on the imaginary axis. Hence, there is no point in $\mathbb{C}$ for which the functions $f(z)$ is differentiable on a neighbourhood (since the Cauchy Riemann equations are necessary for differentiability) and therefore, $f(z)$ is nowhere analytic.
(b) Note that the real and imaginary components of the function

$$f(z) = f(x+iy) = e^y e^{ix} = e^y \cos x + ie^y \sin x$$

are well defined. Hence the function $f(z)$ is defined on all $\mathbb{C}$. If $f(z)$ were analytic then we will have

$$u_x = v_y \Rightarrow -e^y \sin x = e^y \sin x \Rightarrow 2e^y \sin x = 0 \Rightarrow \sin x = 0$$

and

$$u_y = -v_x \Rightarrow e^y \cos x = -e^y \cos x \Rightarrow 2e^y \cos x = 0 \Rightarrow \cos x = 0.$$

On the one hand, we have that the roots of $\sin x$ are $n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, but $\cos(n\pi) = (-1)^n \neq 0$. On the other hand, the roots of $\cos x$ are $(2n-1)\pi/2$ but $\sin((2n-1)\pi/2) = -\cos(n\pi) = -(-1)^n \neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere. ∎

13. Show where the function $z \mapsto x^3 + i(1-y)^3$ is:
   (a) analytic;
   (b) differentiable (here $z = x + iy$).

   Solution. If the function

$$f(z) = f(z+iy) = x^3 + i(1-y)^3$$

were analytic then we will have

$$u_x = v_y \quad \Rightarrow \quad 3x^2 = -3(1-y)^2,$$

which is only satisfied at $x=0,\ y=1$; and, on the other hand,

$$u_y = -v_x \quad \Rightarrow \quad 0=0,$$

which holds everywhere. Note also that the componentes of $f(z)$, and all its first-order partials exist everywhere.
Since the Cauchy-Riemann equations only hold at $z=i$, the function $f(z)$ is only differentiable at $z=i$. Hence, in particular, it is not differentiable on any neighbourhood of any point, and therefore is nowhere analytic. ■

14. Verify that the following functions are analytic on their domain of definition, and state the derivative, (here $z=x+iy=e^{i\theta}$):
 (a) $z \mapsto \ln r + i\theta$, domain $\{z : r>0, 0<\theta<2\pi\}$;
 (b) $z \mapsto \dfrac{4z+1}{z^3-z}$, domain $\mathbb{C}\setminus\{0,1,-1\}$.

Solution. (a) We have $u_r = 1/r$, $v_r=0$, $u_\theta=0$, $v_\theta=1$. Hence u, u_r, u_θ, v, v_r, v_θ are defined and continuous on the domain, with the Cauchy-Riemann equations being satisfied. That is

$$ru_r = v_\theta \quad \Leftrightarrow \quad r\cdot\frac{1}{r}=1, \quad \text{and} \quad u_\theta = -v_r \quad \Leftrightarrow \quad 0=0.$$

The derivative is then

$$f'(z) = e^{-i\theta}(u_r+iv_r) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z},$$

since $z=re^{i\theta}$. This is defined on the whole domain, Therefore the function $f(z)$ is analytic there.
(b) The functions is a rational function and hence exists and is differentiable as long as the denominator does not vanish, which is true on the whole domain. Hence f is analytic on the whole domain, with

$$f'(z) = \frac{(z^3-z)\cdot 4-(4z+1)(3z^2-1)}{(z^3-z)^2} = \frac{-(8z^3+3z+1)}{(z^3-z)^2}.$$

■

2.3.1 Harmonic functions

1. Verify that the following functions u are harmonic, and in each case give a conjugate harmonic function v (*i.e.* v such that $u+iv$ is analytic).
 (a) $u(x,y)=3x^2y+2x^2-y^3-2y^2$,
 (b) $y(x,y)=\ln(x^2+y^2)$.

Solution. (a) If $u(x,y)=3x^2y+2x^2-y^3-2y^2$, then

$$\begin{aligned} u_x &= 6xy+4x, & u_y &= 3x^2-3x^2-4y \\ u_{xx} &= 6y+4, & u_{yy} &= -6y-4 \end{aligned}$$

Thus

$$\Delta u = u_{xx} + u_{yy} = 6y + 4 + (-6y - 4) = 0.$$

Hence, u is harmonic.
The harmonic conjugate of u will satisfy the Cauchy-Riemann equations and have continuos partials of all orders. By Cauchy-Riemann equations

$$u_x = v_y, \quad v_x = -u_y,$$

we have that $v_y = 6xy + 4x$. Thus

$$v = \int (6xy + 4x)dy = 3xy^2 + 4xy + g(x).$$

Thus

$$v_x = 3y^2 + 4y + g'(x)$$

Since $v_x = -u_y$,

$$\begin{aligned} 3y^2 + 4y + g'(x) &= -3x^2 + 3y^2 + 4y \\ g'(x) &= -3x^2 \\ g(x) &= -x^3 \end{aligned}$$

Therefore, the harmonic conjugate is

$$v(x, y) = 3xy^2 + 4xy - x^3.$$

(b) If $u(x, y) = \ln(x^2 - y^2)$, then

$$u_x = \frac{2x}{x^2 + y^2}, \qquad u_y = \frac{2y}{x^2 + y^2}$$
$$u_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}, \qquad u_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Thus

$$\Delta u = u_{xx} + u_{yy} = \frac{2(y^2 - x^2)}{x^2 + y^2} + \frac{2(x^2 - y^2)}{x^2 + y^2} x^2 + y^2 = 0.$$

Hence, u is harmonic.
Similarly to the previous part, by Cauchy-Riemann equations

$$u_x = v_y, \quad v_x = -u_y,$$

we have that $v_y = \dfrac{2x}{x^2 + y^2}$. Thus

$$v = \int \frac{2x}{x^2 + y^2} dy = 2 \arctan \frac{y}{x} + g(x)$$

So we have

$$v_x = \frac{-2y}{x^2 + y^2} + g'(x)$$

Since $v_x = -u_y$,

$$\begin{aligned} \frac{-2y}{x^2+y^2} + g'(x) &= \frac{-2y}{x^2+y^2} \\ g'(x) &= 0 \\ g(x) &= c \quad c \in \mathbb{R} \end{aligned}$$

Hence the harmonic conjugate is

$$v(x,y) = 2\arctan\frac{y}{x}$$

Notice that u is defined on $\mathbb{C}\setminus\{0\}$ and v is not defined if $x=0$. ■

2. Suppose that U solves a Neumann problem for Laplace's equation on a domain Ω. Show that $U+c$ also solves this problem for any $c \in \mathbb{R}$. Does the same result hold for the corresponding Dirichlet problem?

Proof. Let $\mathbf{n}$ be the external unit normal to Ω. If U solves a Neumann problem then U satisfies the following conditions

$$\begin{cases} \Delta U = 0, & \text{in } \Omega; \\ \dfrac{dU}{d\mathbf{n}} = g, & \text{on } \partial\Omega \end{cases}$$

Considering $U+c$, we have

$$\Delta(U+c) = \Delta U = 0.$$

So $U+c$ satisfies Laplace's equation in $\partial\Omega$. Furthermore,

$$\frac{d}{d\mathbf{n}}(U+c) = \nabla(U+c)\cdot\mathbf{n} = \nabla U\cdot\mathbf{n} = \frac{dU}{d\mathbf{n}} = g$$

Then $U+c$ also satisfies the boundary condition on Ω. Therefore, $U+c$ solves the Neumann problem.
On the other hand, if U solves a Dirichlet problem, it satisfies the conditions

$$\begin{cases} \Delta U = 0, & \text{in } \Omega; \\ U = \alpha, & \text{on } \partial\Omega \end{cases}$$

where $\alpha \in \mathbb{R}$. Certainly, $U+c$ satisfies Laplace's equation for any $c \in \mathbb{R}$. However, $U+c$ satisfies the second condition only when $c=0$. Hence, $U+c$ does not solve a Dirichlet problem. ■

3. Let Λ be the domain $\{w \mid \operatorname{Im} w < \pi\}$. Denote the two components of the boundary of Λ by $\Gamma_1 = \{w \mid \operatorname{Im} w = 0\}$ and $\Gamma_2 = \{w \mid \operatorname{Im} w = \pi\}$. Let C be an arbitrary real constant.

(a) Verify that the function

$$T = \mathrm{Im}\left(\frac{w}{\pi} + C\cosh w\right)$$

is harmonic on Λ, and satisfies the Dirichlet boundary conditions

$$T\Big|_{\Gamma_1} = 0, \quad T\Big|_{\Gamma_2} = 1.$$

(b) For what values (if any) of C is T a bounded function on Λ, *i.e.* for what values of C does there exist an $M > 0$ such that $|T(w)| \leq M$ for all $w \in \Lambda$?

Solution. (a) Let $w = u + iv$, then we have

$$T = \mathrm{Im}\left(\frac{w}{\pi} + C\cosh w\right) = \frac{v}{\pi} + C\sinh u \sin v.$$

Then

$$\begin{aligned} T_u &= C\cosh u \sin v, \qquad & T_v &= \frac{1}{\pi} + C\sinh u \cos v \\ T_{uu} &= C\sinh u \sin v, \qquad & T_{vv} &= -C\sinh u \sin v \end{aligned}$$

Thus

$$\Delta T = T_{uu} + T_{vv} = C\sinh u \sin v + (-C\sinh u \sin v) = 0.$$

Hence, T is harmonic on Λ.
Notice that on Γ_1, we have that $v = 0$ and so $T = 0$. Finally, on Γ_2 we have that $v = \pi$, then $T = \dfrac{\pi}{\pi} + 0 = 1$.
(b) If $C = 0$, the solution is $\dfrac{v}{\pi}$ and $|T| < 1$ on Λ. If $C \neq 0$, considering the point $w_0 = \mu + \dfrac{\pi}{2}i$, we have

$$\begin{aligned} T(w_0) &= \frac{1}{2} + C\sinh\mu \\ &= \frac{1}{2}\left(1 + Ce^{\mu} + Ce^{-\mu}\right) \end{aligned}$$

For a function μ sufficiently large

$$|T(w_0)| > \frac{|C|}{2}e^{\mu},$$

which tends to ∞ as $\mu \to \infty$. Therefore, T is unbounded on Λ. ■

3. Complex Integrals

3.1 Contour integrals

1. Evaluate the following integrals:

   (a) $\int_1^2 \left(t^2+i\right)^2 dt$;

   (b) $\int_0^{\pi/4} e^{-2it} dt$;

   (c) $\int_0^{\infty} te^{zt} dt$ when $\text{Re}(z) < 0$.

   Solution.

   (a)

$$\int_1^2 \left(t^2+i\right)^2 dt = \int_1^2 (t^4+2it^2+i^2)dt$$
$$= \frac{t^5}{5} + \frac{2it^3}{3} - t\Big|_1^2 = \frac{26}{5} + \frac{14}{3}i$$

   (b)

$$\int_0^{\pi/4} e^{-2it} dt = -\left.\frac{e^{-2it}}{2i}\right|_0^{\pi/4}$$
$$= \frac{1+i}{2i} = \frac{1}{2} - \frac{i}{2}$$

(c)

$$\begin{aligned}
\int_0^\infty te^{zt}dt &= \frac{1}{z}\int_0^\infty t\,d(e^{zt}) \\
&= \frac{1}{z}\left(te^{zt}\Big|_0^\infty - \int_0^\infty e^{zt}dt\right) \\
&= \frac{1}{z}\left(\lim_{t\to\infty} te^{zt} - \frac{1}{z}\left(\lim_{t\to\infty} e^{zt} - 1\right)\right) \\
&= \frac{1}{z^2}
\end{aligned}$$

where

$$\lim_{t\to\infty} te^{zt} = \lim_{t\to\infty} e^{zt} = 0$$

because

$$\lim_{t\to\infty} |te^{zt}| = \lim_{t\to\infty} te^{t\,\mathrm{Re}(z)} = \lim_{t\to\infty} \frac{t}{e^{-xt}} = -\lim_{t\to\infty} \frac{1}{xe^{-xt}} = 0$$

by L'Hospital (see Problem 8 in section 2.2), and

$$\lim_{t\to\infty} |e^{zt}| = \lim_{t\to\infty} e^{t\,\mathrm{Re}(z)} = \lim_{t\to\infty} \frac{1}{e^{-xt}} = 0$$

as $x = \mathrm{Re}(z) < 0$.

2. Find the contour integral $\int_\gamma \bar{z}dz$ for
   (a) γ is the triangle ABC oriented counterclockwise, where $A = 0$, $B = 1+i$ and $C = -2$;
   (b) γ is the circle $|z-i| = 2$ oriented counterclockwise.

Solution. (a)

$$\begin{aligned}
\int_\gamma \bar{z}dz &= \int_{AB} \bar{z}dz + \int_{BC} \bar{z}dz + \int_{CA} \bar{z}dz \\
&= \int_0^1 \overline{t(1+i)}\,d(t(1+i)) \\
&\quad + \int_0^1 \overline{(1-t)(1+i)-2t}\,d((1-t)(1+i)-2t) \\
&\quad + \int_0^1 \overline{-2(1-t)}\,d(-2(1-t)) \\
&= \int_0^1 2t\,dt + \int_0^1 ((2i-4)+10t)dt + \int_0^1 4(t-1)dt \\
&= 1 + (2i-4) + 5 - 2 = 2i
\end{aligned}$$

(b)

$$\int_0^{2\pi} \overline{i+2e^{it}}\,d(i+2e^{it}) = \int_0^{2\pi} 2i(-i+2e^{-it})e^{it}dt = 8\pi i.$$

■

3. Compute the following contour integral

$$\int_L \bar{z}dz,$$

where L is the boundary of the triangle ABC with $A = 0$, $B = 1$ and $C = i$, oriented counter-clockwise.

Solution.

$$\begin{aligned}
\int_L \bar{z}dz &= \int_{AB} \bar{z}dz + \int_{BC} \bar{z}dz + \int_{CA} \bar{z}dz \\
&= \int_0^1 \bar{t}dt + \int_0^1 \overline{(1-t)+ti}\,d((1-t)+ti) \\
&\quad + \int_0^1 \overline{(1-t)i}\,d((1-t)i) \\
&= \int_0^1 tdt + (-1+i)\int_0^1 ((1-t)-ti)dt - \int_0^1 (1-t)dt = i
\end{aligned}$$

■

4. Evaluate the contour integral

$$\int_C f(z)dz$$

using the parametric representations for C, where

$$f(z) = \frac{z^2-1}{z}$$

and the curve C is
(a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;
(b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
(c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Solution.
(a)

$$\int_C f(z)dz = \int_0^{\pi} \frac{4e^{2i\theta}-1}{2e^{i\theta}} d(2e^{i\theta}) = (2e^{2i\theta} - i)\big|_0^{\pi} = -\pi i$$

(b)

$$\int_C f(z)dz = \int_{\pi}^{2\pi} \frac{4e^{2i\theta}-1}{2e^{i\theta}} d(2e^{i\theta}) = (2e^{2i\theta} - i)\big|_{\pi}^{2\pi} = -\pi i$$

(c) Adding (a) and (b), we have $-2\pi i$.

5. Redo previous Problem 4 using an antiderivative of $f(z)$.

Solution. For (a),

$$\begin{aligned}\int_C f(z)dz &= \frac{z^2}{2}\Big|_2^{-2} - \left(\lim_{\substack{z\to -2\\ \mathrm{Im}(z)>0}} \mathrm{Log}(z) - \mathrm{Log}(2)\right)\\ &= -(\ln 2 + \pi i - \ln 2) = -\pi i.\end{aligned}$$

For (b),

$$\begin{aligned}\int_C f(z)dz &= \frac{z^2}{2}\Big|_{-2}^{2} - \left(\mathrm{Log}(2) - \lim_{\substack{z\to -2\\ \mathrm{Im}(z)<0}} \mathrm{Log}(z)\right)\\ &= -(\ln 2 - (\ln 2 - \pi i)) = -\pi i.\end{aligned}$$

6. Let C_R be the circle $|z| = R$ ($R > 1$) oriented counterclockwise. Show that

$$\left|\int_{C_R} \frac{\mathrm{Log}(z^2)}{z^2} dz\right| < 4\pi\left(\frac{\pi + \ln R}{R}\right)$$

and then

$$\lim_{R\to\infty} \int_{C_R} \frac{\mathrm{Log}(z^2)}{z^2} dz = 0.$$

Proof. Using expression (1.1) in Problem 5, we have

$$|\mathrm{Log}(z^2)| \le \left|\ln |z^2|\right| + \pi = 2\ln R + \pi$$

for $|z| = R > 1$. Therefore,

$$\begin{aligned}\left|\int_{C_R} \frac{\mathrm{Log}(z^2)}{z^2} dz\right| &\le 2\pi R\left(\frac{\pi + 2\ln R}{R^2}\right)\\ &= 4\pi\left(\frac{\pi/2 + \ln R}{R}\right) < 4\pi\left(\frac{\pi + \ln R}{R}\right).\end{aligned}$$

And since

$$\lim_{R\to\infty} 4\pi\left(\frac{\pi + \ln R}{R}\right) = 4\pi \lim_{R\to\infty} \frac{1}{R} = 0$$

by L'Hospital (see Problem 8 in section 2.2),

$$\lim_{R\to\infty} \int_{C_R} \frac{\mathrm{Log}(z^2)}{z^2} dz = 0.$$

∎

7. Without evaluating the integral, show that
$$\left|\int_C \frac{dz}{\bar{z}^2+\bar{z}+1}\right| \le \frac{9\pi}{16}$$
where C is the arc of the circle $|z|=3$ from $z=3$ to $z=3i$ lying in the first quadrant.

Proof. Since
$$|\bar{z}^2+\bar{z}+1| \ge |\bar{z}^2| - |\bar{z}| - 1 = |z|^2 - |z| - 1 = 5$$
for $|z|=3$,
$$\left|\frac{1}{\bar{z}^2+\bar{z}+1}\right| \le \frac{1}{5}.$$
Therefore,
$$\left|\int_C \frac{dz}{\bar{z}^2+\bar{z}+1}\right| \le \frac{6\pi}{4}\left(\frac{1}{5}\right) = \frac{3\pi}{10} < \frac{9\pi}{16}.$$
■

8. Evaluate the integral $\int_C \operatorname{Re}(z)dz$ for the following contours C from -4 to 4:
   (a) The line segments from -4 to $-4-4i$ to $4-4i$ to 4;
   (b) the lower half of the circle with radius 4, centre 0;
   (c) the upper half of the circle with radius 4, centre 0.
   (d) What conclusions (if any) can you draw about the function $f(z)=\operatorname{Re}(z)$ from this?

Solution. (a) Notice that the contour C consists of three contours:
   i. C_1 defined by $z(t) = -4-4it$, with $0 \le t \le 1$, followed by
   ii. C_2 defined by $z(t) = -4(1-2t)-4i$, with $0 \le t \le 1$, and finally
   iii. C_3 defined by $z(t) = 4-4i(1-t)$, with $0 \le t \le 1$

Thus
$$\int_C f(z)dz = \int_{C_1+C_2+C_3} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz \qquad (3.1)$$
where $f(z)=\operatorname{Re}(z)$. Recall also that $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$.
Now, for C_1 we have that $z'(t) = -4i$. Then
$$\int_{C_1} f(z)dz = \int_0^1 (-4)(-4i)dt = 16i\int_0^1 dt = 16it\Big|_0^1 = 16i.$$
For C_2 we have that $z'(t) = 8$, then
$$\begin{aligned}\int_{C_2} f(z)dz &= \int_0^1 [-4(1-2t)](8)dt \\ &= \int_0^1 (64t-32)dt \\ &= 64\int_0^1 tdt - 32\int_0^1 dt \\ &= 64\frac{t^2}{2}\Big|_0^1 - 32t\Big|_0^1 \\ &= 32-32=0\end{aligned}$$

Finally for C_3 we have that $z'(t) = 4i$, then

$$\int_{C_3} f(z)dz = \int_0^1 (4)(4i)dt = 16i\int_0^1 dt = 16it\Big|_0^1 = 16i.$$

Therefore, using expression 3.1, we obtain

$$\int_C \mathrm{Re}(z)dz = 32i$$

(b) In this case, the contour C is defined by

$$z(t) = 4e^{it} = 4\cos t + 4i\sin t,$$

with $\pi \le t \le 2\pi$. Here we have $z'(t) = 4ie^{it}$.
Thus

$$\begin{aligned}
\int_C \mathrm{Re}(z)dz &= \int_\pi^{2\pi} (4\cos t)(4ie^{it})dt \\
&= 16i\int_\pi^{2\pi} \cos t e^{it} dt \\
&= 16i\int_\pi^{2\pi} \cos^2 t dt - 16\int_\pi^{2\pi} \cos t \sin t dt \\
&= 8i\int_\pi^{2\pi} [1+\cos(2t)]dt - 16\cdot\frac{\sin^2 t}{2}\Big|_\pi^{2\pi} \\
&= 8i\left[t+\frac{\sin(2t)}{2}\right]\Big|_\pi^{2\pi} - 0 = 8\pi i
\end{aligned}$$

(c) Finally, in this case, the contour C is defined by

$$z(t) = -4e^{-it} = -4\cos t + 4i\sin t,$$

with $0 \le t \le \pi$. Here we have $z'(t) = 4ie^{-it}$.
Thus

$$\begin{aligned}
\int_C \mathrm{Re}(z)dz &= \int_0^{\pi} (-4\cos t)(4ie^{-it})dt \\
&= -16i\int_0^{\pi} \cos t e^{-it} dt \\
&= -16i\int_0^{\pi} \cos^2 t dt + 16\int_0^{\pi} \cos t \sin t dt \\
&= -8i\int_0^{\pi} [1+\cos(2t)]dt + 16\cdot\frac{\sin^2 t}{2}\Big|_0^{\pi} \\
&= -8i\left[t+\frac{\sin(2t)}{2}\right]\Big|_0^{\pi} + 0 = -8\pi i
\end{aligned}$$

(d) We have seen that the integral along each contour has a different value.
The reason is that the function $f(z) = \mathrm{Re}(z)$ is not analytic on any domain containing any of the contours discussed in parts (a), (b) and (c). In fact, this function is nowhere analytic.

■

3.2 Cauchy Integral Theorem and Cauchy Integral Formula

1. Evaluate the following integrals, justifying your procedures. For c) and d) you should also state why the integral is well defined (i.e., independent of the path taken).
   (a) $\int_C \frac{2dz}{z^2-1}$, where C is the circle with radius 1/2, centre 1, positively oriented;
   (b) $\int_C \left(e^z + \frac{1}{z}\right) dz$, where C is the lower half of the circle with radius 1, centre 0, negatively oriented;
   (c) $\int_C ze^{z^2} dz$;
   (d) $\int_C \cosh z dz$.

Solution. (a) Notice that
$$\frac{2}{z^2-1} = \frac{1}{z-1} - \frac{1}{z+1}.$$
Thus
$$\int_C \frac{2}{z^2-1} dz = \int_C \frac{1}{z-1} dz - \int_C \frac{1}{z+1} dz.$$
On the one hand,
$$\int_C \frac{1}{z-1} dz = 2\pi i$$
by Cauchy integral formula, $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0} dz$.

On the other hand, since $1/(z+1)$ is analytic on and inside C, then
$$\int_C \frac{1}{z+1} dz = 0$$
by Cauchy's Theorem. Therefore,
$$\int_C \frac{2}{z^2-1} dz = 2\pi i.$$

(b) Notice that the integrand $f(z) = e^z - 1/z$ is analytic on C. The function
$$F(z) = e^z - \text{Log}\, z$$
serves as an antiderivative of $f(z)$. Here $\text{Log}\, z$ is a branch of the logarithm chosen with the branch cut on the positive imaginary axis. That is,
$$\text{Log}\, z = \ln r + i\theta, \qquad \left(r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}\right).$$
Thus
$$\int_C \left(e^z - \frac{1}{z}\right) dz = (e^z - \text{Log}\, z)\Big|_1^{-1} = \frac{1}{e} - \pi i - e + 2\pi i = \frac{1}{e} - e + \pi i.$$

(c) Since the integrand $f(z) = ze^{z^2}$ is analytic, the integral is path independent. An antiderivative of $f(z)$ is

$$F(z) = \frac{e^{z^2}}{2}.$$

Thus

$$\int_C ze^{z^2}\,dz = \left.\frac{e^{z^2}}{2}\right|_0^i = \frac{1}{2e} - \frac{1}{2}.$$

(d) Since the integrand $f(z) = \cosh z$ is analytic, the integral is path independent. An antiderivative of $f(z)$ is

$$F(z) = \sinh z.$$

Thus

$$\int_C \cosh z\,dz = \sinh z\Big|_{\pi i}^{2\pi i} = \sinh(2\pi i) - \sinh(\pi i) = 0.$$

■

2. Let C_R be the circle with radius R, centre 0, positively oriented. Show that

$$\lim_{R\to\infty}\int_{C_R} \frac{z^2+4z+7}{(z^2+4)(z^2+2z+2)}dz = 0.$$

Use this fact to prove that

$$\int_C \frac{z^2+4z+7}{(z^2+4)(z^2+2z+2)}dz = 0.$$

where C is the circle with radius 5, centre 2, positively oriented.

Solution. Recall that for a contour C of length L and a piecewise continuous $f(z)$ on C, if M is a nonnegative constant such that $|f(z)| < M$ for all points z on C at which $f(z)$ is defined, then

$$\left|\int_C f(z)dz\right| < ML.$$

Now, consider the function

$$f(z) = \frac{z^2+4z+7}{(z^2+4)(z^2+2z+2)}.$$

For $|z|$ large, we have that

$$|f(z)| = \frac{\left|1+\frac{4}{z}+\frac{7}{z^2}\right|}{\left|1+\frac{4}{z^2}\right|\,|z^2+2z+2|} \approx \frac{1}{|z|^2}$$

On the circle C_R we have that $|z| = R$. Thus, for R large we have that

$$|f(z)| < \frac{2}{R^2}$$

Since the length of C_R is $2\pi R$, then

$$\left| \int_{C_R} f(z)dz \right| < ML = \frac{2}{R^2} 2\pi R = \frac{4\pi}{R}$$

which tends to 0 as $R \to \infty$.
On the other hand, notice that he singularities

$$z_1 = 2i, \;\; z_2 = -2i, \;\; z_3 = -1+i, \quad \text{and} \quad z_4 = -1-i$$

of the function

$$f(z) = \frac{z^2+4z+7}{(z^2+4)(z^2+2z+2)}$$

are inside C and since $f(z)$ is analytic on the annulus defined by C and C_R with $R > 7$, then

$$\int_C f(z)dz = \int_{C_R} f(z)dz.$$

Thus

$$\lim_{R\to\infty} \int_C f(z)dz = \lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

Hence

$$\int_C f(z)dz = 0.$$

■

3. Evaluate

$$\int_C \frac{\sin z}{(z+1)^7} dz,$$

where C is the circle of radius 5, centre 0, positively oriented.

Solution. Recall the extension of the Cauchy integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Considering the function $f(z) = \sin z$, which is analytic on $\mathbb{C}$, we have

$$f^{(6)}(-1) = \frac{6!}{2\pi i} \int_C \frac{\sin z}{(z-(-1))^{6+1}} = \frac{6!}{2\pi i} \int_C \frac{\sin z}{(z+1)^7}.$$

Since $f^{(6)}(z) = -\sin z$, then

$$\int_C \frac{\sin z}{(z+1)^7} = \int_C \frac{\sin z}{(z-(-1))^{6+1}} = -\frac{2\pi i}{6!} \sin(-1) = \frac{2\pi \sin(1)}{6!} i$$

■

4. Let C be the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented counterclockwise. Compute the contour integral

$$\int_C (e^z - \overline{z})dz.$$

Solution. By Cauchy Integral Theorem, $\int_C e^z dz = 0$ since C is closed and e^z is entire. Therefore,

$$\begin{aligned}\int_C (e^z - \overline{z})dz &= -\int_C \overline{z}dz = -\int_{\overline{p_1p_2}} \overline{z}dz - \int_{\overline{p_2p_3}} \overline{z}dz - \int_{\overline{p_3p_1}} \overline{z}dz \\ &= -\int_0^1 (-3it)d(3it) - \int_0^1 (-3i(1-t)-4t)d(3i(1-t)-4t) \\ &\quad - \int_0^1 (-4)(1-t)d((-4)(1-t)) \\ &= -\frac{9}{2} - \frac{7}{2} - 12i + 8 = -12i\end{aligned}$$

where $p_1 = 0$, $p_2 = 3i$ and $p_3 = -4$. ■

5. Compute

$$\int_{-1}^1 z^i dz$$

where the integrand denote the principal branch

$$z^i = \exp(i \operatorname{Log} z)$$

of z^i and where the path of integration is any continuous curve from $z = -1$ to $z = 1$ that, except for its starting and ending points, lies below the real axis.

Solution. Note that $z^{i+1}/(i+1)$ is an anti-derivative of z^i outside the branch locus $(-\infty, 0]$. So

$$\begin{aligned}\int_{-1}^1 z^i dz &= \left.\frac{z^{i+1}}{i+1}\right|_1 - \lim_{\substack{z\to -1 \\ \operatorname{Im}(z)<0}} \frac{z^{i+1}}{i+1} \\ &= \frac{1}{i+1} - \frac{\exp((i+1)(-\pi i))}{i+1} \\ &= \frac{1+e^\pi}{i+1} = \frac{1+e^\pi}{2}(1-i)\end{aligned}$$

■

6. Apply Cauchy Integral Theorem to show that

$$\int_C f(z)dz = 0$$

when C is the unit circle $|z| = 1$, in either direction, and when
(a) $f(z) = \dfrac{z^3}{z^2+5z+6}$;
(b) $f(z) = e^{\tan z}$;
(c) $f(z) = \mathrm{Log}(z+3i)$.

Solution. By Cauchy Integral Theorem, $\int_{|z|=1} f(z)dz = 0$ if $f(z)$ is analytic on and inside the circle $|z| = 1$. Hence it is enough to show that $f(z)$ is analytic in $\{|z| \le 1\}$.
(a) $f(z)$ is analytic in $\{z \ne -2, -3\}$ and hence analytic in $\{|z| \le 1\}$.
(b) $f(z)$ is analytic in $\{z : \cos z = 0\} = \{z = n\pi + \pi/2, n \in \mathbb{Z}\}$. Since $|n\pi + \pi/2| > 1$ for all integers n, $f(z)$ is analytic in $\{|z| \le 1\}$.
(c) $\mathrm{Log}(z)$ is analytic in $\mathbb{C}\backslash(-\infty, 0]$ and hence $\mathrm{Log}(z+3i)$ is analytic in $\mathbb{C}\backslash\{z : z = x - 3i, x \in (-\infty, 0]\}$. Since $|x - 3i| > 1$ for all x real, $f(z)$ is analytic in $\{|z| \le 1\}$.

■

7. Let C_1 denote the positively oriented boundary of the curve given by $|x| + |y| = 2$ and C_2 be the positively oriented circle $|z| = 4$. Apply Cauchy Integral Theorem to show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

when
(a) $f(z) = \dfrac{z+1}{z^2+1}$;
(b) $f(z) = \dfrac{z+2}{\sin(z/2)}$;
(c) $f(z) = \dfrac{\sin(z)}{z^2+6z+5}$.

Solution. By Cauchy Integral Theorem, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ if $f(z)$ is analytic on and between C_1 and C_2. Hence it is enough to show that $f(z)$ is analytic in $\{|x| + |y| \ge 2, |z| \le 4\}$.
(a) $f(z)$ is analytic in $\{z \ne \pm i\}$. Since $\pm i \in \{|x| + |y| < 2\}$, $f(z)$ is analytic in $\{|x| + |y| \ge 2, |z| \le 4\}$.
(b) $f(z)$ is analytic in $\{z : \sin(z/2) \ne 0\} = \{z \ne 2n\pi : n \in \mathbb{Z}\}$. Since $2n\pi \in \{|x| + |y| < 2\}$ for $n = 0$ and $|2n\pi| > 4$ for $n \ne 0$ and $n \in \mathbb{Z}$, $f(z)$ is analytic in $\{|x| + |y| \ge 2, |z| \le 4\}$.
(c) $f(z)$ is analytic in $\{z \ne -1, -5\}$. Since $-1 \in \{|x| + |y| < 2\}$ for $n = 0$ and $|-5| > 4$, $f(z)$ is analytic in $\{|x| + |y| \ge 2, |z| \le 4\}$.

■

8. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals
   (a) $\displaystyle\int_C \frac{zdz}{z+1}$;
   (b) $\displaystyle\int_C \frac{\cosh z}{z^2+z}dz$;
   (c) $\displaystyle\int_C \frac{\tan(z/2)}{z-\pi/2}dz$.

Solution. (a) By Cauchy Integral Formula,

$$\int_C \frac{zdz}{z+1} = 2\pi i(-1) = -2\pi i.$$

(b) By Cauchy Integral Theorem,

$$\int_C \frac{\cosh z}{z^2+z}dz = \int_{|z|=r} \frac{\cosh z}{z^2+z}dz + \int_{|z+1|=r} \frac{\cosh z}{z^2+z}dz$$

for $r = 1/2$. By Cauchy Integral Formula,

$$\int_{|z|=r} \frac{\cosh z}{z^2+z}dz = 2\pi i \left.\frac{\cosh(z)}{z+1}\right|_{z=0} = 2\pi i$$

and

$$\int_{|z+1|=r} \frac{\cosh z}{z^2+z}dz = 2\pi i \left.\frac{\cosh z}{z}\right|_{z=-1} = -2\pi i \cosh(-1).$$

Hence

$$\int_C \frac{\cosh z}{z^2+z}dz = 2\pi i(1-\cosh(-1)).$$

(c) Note that $\tan(z/2)$ is analytic in $\{z \neq (2n+1)\pi : n \in \mathbb{Z}\}$ and hence analytic inside C. Therefore,

$$\int_C \frac{\tan(z/2)}{z-\pi/2}dz = 2\pi i \tan(\pi/4) = 2\pi i$$

by Cauchy Integral Formula.

■

9. Find the value of the integral $g(z)$ around the circle $|z-i| = 2$ oriented counterclockwise when
   (a) $g(z) = \dfrac{1}{z^2+4}$;
   (b) $g(z) = \dfrac{1}{z(z^2+4)}$.

Solution. (a) Since $-2i \notin \{|z-i| \le 2\}$ and $2i \in \{|z-i| \le 2\}$,

$$\int_{|z-i|=2} g(z)dz = \int_{|z-i|=2} \frac{(z+2i)^{-1}}{z-2i} dz = 2\pi i(2i+2i)^{-1} = \frac{\pi}{2}$$

by Cauchy Integral Formula.

(b) By Cauchy Integral Theorem,

$$\int_{|z-i|=2} g(z)dz = \int_{|z|=r} g(z)dz + \int_{|z-2i|=r} g(z)dz$$

for $r < 1/2$. Since

$$\int_{|z|=r} g(z)dz = 2\pi i \left.\frac{1}{z^2+4}\right|_{z=0} = \frac{\pi i}{2}$$

and

$$\int_{|z-2i|=r} g(z)dz = 2\pi i \left.\frac{1}{z(z+2i)}\right|_{z=2i} = -\frac{\pi i}{4}$$

by Cauchy Integral Formula,

$$\int_{|z-i|=2} g(z)dz = \frac{\pi i}{4}$$

■

10. Compute the integrals of the following functions along the curves $C_1 = \{|z| = 1\}$ and $C_2 = \{|z-2| = 1\}$, both oriented counterclockwise:

(a) $\dfrac{1}{2z-z^2}$;

(b) $\dfrac{\sinh z}{(2z-z^2)^2}$.

Solution. (a)

$$\int_{|z|=1} \frac{dz}{2z-z^2} = \int_{|z|=1} \frac{(2-z)^{-1}}{z} dz = 2\pi i(2-0)^{-1} = \pi i$$

(b)

$$\begin{aligned}\int_{|z|=1} \frac{\sinh z}{(2z-z^2)^2} dz &= \int_{|z|=1} \frac{(\sinh z)(2-z)^{-2}}{z^2} dz \\ &= 2\pi i((\sinh z)(2-z)^{-2})'\Big|_{z=0} = \frac{\pi i}{2}\end{aligned}$$

■

11. Show that if f is analytic inside and on a simple closed curve C and z_0 is not on C, then

$$(n-1)!\int_C \frac{f^{(m)}(z)}{(z-z_0)^n}dz = (m+n-1)!\int_C \frac{f(z)}{(z-z_0)^{m+n}}dz$$

for all positive integers m and n.

Proof. If z_0 lies outside C, then

$$\int_C \frac{f^{(m)}(z)}{(z-z_0)^n}dz = \int_C \frac{f(z)}{(z-z_0)^{m+n}}dz = 0$$

by Cauchy Integral Theorem, since $f^{(m)}z/(z-z_0)^n$ and $f(z)/(z-z_0)^{m+n}$ are analytic on and inside C.
If z_0 lies inside C, then

$$(n-1)!\int_C \frac{f^{(m)}(z)}{(z-z_0)^n}dz = (f^{(m)}(z))^{(n-1)}\big|_{z=z_0} = f^{(m+n-1)}(z_0)$$

and

$$(m+n-1)!\int_C \frac{f(z)}{(z-z_0)^{m+n}}dz = f^{(m+n-1)}(z_0)$$

by Cauchy Integral Formula. Therefore,

$$(n-1)!\int_C \frac{f^{(m)}(z)}{(z-z_0)^n}dz = (m+n-1)!\int_C \frac{f(z)}{(z-z_0)^{m+n}}dz.$$

■

12. Let $f(z)$ be an entire function. Show that $f(z)$ is a constant if $|f(z)| \le \ln(|z|+1)$ for all $z \in \mathbb{C}$.

Proof. For every $z_0 \in \mathbb{C}$, we have

$$f'(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}dz$$

for all $R > 0$. Since

$$\left|\frac{f(z)}{(z-z_0)^2}\right| \le \frac{\ln(|z|+1)}{R^2} \le \frac{\ln(R+|z_0|+1)}{R^2}$$

for $|z-z_0| = R$,

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}dz\right| \le \frac{\ln(R+|z_0|)+1}{R}.$$

And since

$$\lim_{R\to\infty} \frac{\ln(R+|z_0|+1)}{R} = \lim_{R\to\infty} \frac{1}{R+|z_0|+1} = 0,$$

by L'Hospital (see Problem 8, section 2.2), we conclude that $|f'(z_0)| = 0$ and hence $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$. Therefore, $f(z)$ is a constant. ■

13. Let C_N be the boundary of the square

$$\{|x| \le N\pi,\ |y| \le N\pi\},$$

where N is a positive integer. Show that

$$\lim_{N\to\infty} \int_{C_N} \frac{dz}{z^3 \cos z} = 0$$

Proof. When $z = x + yi \in C_N$, either $x = \pm N\pi$ or $y = \pm N\pi$. When $x = \pm N\pi$,

$$|\cos z|^2 = (\cos x)^2 + (\sinh y)^2 \ge (\cos x)^2 = (\cos(N\pi))^2 = 1$$

When $y = \pm N\pi$,

$$|\cos z|^2 = (\cos x)^2 + (\sinh y)^2 \ge (\sinh y)^2 = (\sinh(N\pi))^2 > 1$$

Therefore, $|\cos z| \ge 1$ when $z \in C_N$. We also have $|z| \ge N\pi$ when $z \in C_N$. Therefore,

$$\left|\frac{1}{z^3 \cos z}\right| \le \frac{1}{N^3\pi^3}$$

and

$$\left|\int_{C_N} \frac{dz}{z^3 \cos z}\right| \le \frac{1}{N^3\pi^3} \int_{C_N} |dz| = \frac{8N\pi}{N^3\pi^3} = \frac{8}{N^2\pi^2}$$

Since

$$\lim_{N\to\infty} \frac{8}{N^2\pi^2} = 0$$

we conclude

$$\lim_{N\to\infty} \int_{C_N} \frac{dz}{z^3 \cos z} = 0$$

■

14. Let C_N be the boundary of the square

$$\left\{|x| \le N\pi + \frac{\pi}{2},\ |y| \le N\pi + \frac{\pi}{2}\right\}$$

oriented counterclockwise, where N is a positive integer. Show that

$$\lim_{N\to\infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0.$$

[Refer to: problem 6 in section 4.3]

Proof. When $z = x + yi \in C_N$, either $x = \pm(N\pi + \pi/2)$ or $y = \pm(N\pi + \pi/2)$. When $x = \pm(N\pi + \pi/2)$,

$$|\sin z|^2 = (\sin x)^2 + (\sinh y)^2 \geq (\sin x)^2 = (\sin(N\pi + \pi/2))^2 = 1$$

When $y = \pm(N\pi + \pi/2)$,

$$\begin{aligned} |\sin z|^2 &= (\sin x)^2 + (\sinh y)^2 \geq (\sinh y)^2 \\ &= (\sinh(N\pi + \pi/2))^2 \geq (\sinh(3\pi/2))^2 > 1. \end{aligned}$$

Therefore, $|\sin z| \geq 1$ when $z \in C_N$. We also have $|z| \geq N\pi + \pi/2$ when $z \in C_N$. Consequently,

$$\left|\frac{1}{z^2 \sin z}\right| \leq \frac{1}{(N+1/2)^2\pi^2}$$

and

$$\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \leq \frac{1}{(N+1/2)^2\pi^2} \int_{C_N} |dz| = \frac{8(N+1/2)\pi}{(N+1/2)^2\pi^2} = \frac{8}{(N+1/2)\pi}.$$

And since

$$\lim_{N\to\infty} \frac{8}{(N+1/2)\pi} = 0$$

we conclude

$$\lim_{N\to\infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0$$

■

15. Compute the contour integral

$$\int_C \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz,$$

where C is the circle $|z| = 2$ oriented counter-clockwise.

Solution. First, we show that all roots of

$$z^{2011} + z^{2010} + z^{2009} + 1 = 0$$

lie inside $|z| < 2$. Otherwise, suppose that

$$z^{2011} + z^{2010} + z^{2009} + 1 = 0$$

for some $|z| \geq 2$. Then

$$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^{2011}} = 0$$

and hence

$$1 = \left| -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^{2011}} \right| \leq \frac{1}{|z|} + \frac{1}{|z|^2} + \frac{1}{|z|^{2011}}.$$

When $|z| \geq 2$,

$$\frac{1}{|z|} + \frac{1}{|z|^2} + \frac{1}{|z|^{2011}} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{2^{2011}} < 1.$$

This is a contradiction. Therefore, all roots of

$$z^{2011} + z^{2010} + z^{2009} + 1 = 0$$

lie inside $|z| < 2$. It follows that

$$\int_C \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz = \int_{|z|=R} \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz$$

for all $R > 2$ by CIT.
We have

$$\frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} = 1 - \frac{1}{z} + \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)}.$$

Since

$$\left| \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)} \right| \leq \frac{R^{2009} + R + 1}{R(R^{2011} - R^{2010} - R^{2009} - 1)}$$

for $|z| = R$,

$$\left| \int_{|z|=R} \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)} dz \right| \leq \frac{2\pi(R^{2009} + R + 1)}{R^{2011} - R^{2010} - R^{2009} - 1}$$

and hence

$$\lim_{R \to \infty} \int_{|z|=R} \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)} dz = 0.$$

And we have

$$\int_{|z|=R} dz = 0 \text{ and } \int_{|z|=R} \frac{dz}{z} = 2\pi i.$$

Therefore,

$$\int_C \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz = -2\pi i.$$

■

16. Calculate

$$\int_C \frac{z^{2008}}{z^{2009}+z+1}dz$$

where C is the circle $|z|=2$ oriented counter-clockwise.

Solution. First, we prove that all zeroes of $z^{2009}+z+1$ lie inside the circle $|z|=2$. Otherwise, $z^{2009}+z+1=0$ for some $|z|\ge 2$. Then

$$z^{2009}+z+1=0 \Rightarrow 1+\frac{1}{z^{2008}}+\frac{1}{z^{2009}}=0$$

On the other hand,

$$\left|1+\frac{1}{z^{2008}}+\frac{1}{z^{2009}}\right| \ge 1-\frac{1}{|z|^{2008}}-\frac{1}{|z|^{2009}} \ge 1-\frac{1}{2^{2008}}-\frac{1}{2^{2009}}>0$$

for $|z|\ge 2$. Contradiction. So all zeroes of $z^{2009}+z+1$ lie inside the circle $|z|=2$ and hence $z^{2008}/(z^{2009}+z+1)$ is analytic in $|z|\ge 2$. Therefore,

$$\int_{|z|=2}\frac{z^{2008}}{z^{2009}+z+1}dz=\int_{|z|=R}\frac{z^{2008}}{z^{2009}+z+1}dz$$

for all $R>2$ by Cauchy Integral Theorem.
We observe that

$$\frac{z^{2008}}{z^{2009}+z+1}-\frac{1}{z}=-\frac{z+1}{z(z^{2009}+z+1)}.$$

For $|z|=R>2$,

$$\left|\frac{z+1}{z(z^{2009}+z+1)}\right| \le \frac{R+1}{R(R^{2009}-R-1)}$$

and hence

$$\left|\int_{|z|=R}\frac{z+1}{z^{2009}+z+1}dz\right| \le \frac{2\pi(R+1)}{R^{2009}-R-1}.$$

It follows that

$$\lim_{R\to\infty}\int_{|z|=R}\frac{z+1}{z^{2009}+z+1}dz=0.$$

Therefore,

$$\begin{aligned}\int_{|z|=2}\frac{z^{2008}}{z^{2009}+z+1}dz &= \lim_{R\to\infty}\int_{|z|=R}\frac{z^{2008}}{z^{2009}+z+1}dz\\ &= \lim_{R\to\infty}\int_{|z|=R}\frac{dz}{z}=2\pi i.\end{aligned}$$

■

17. Let C be the circle $|z| = 1$ oriented counter-clockwise.

 (a) Compute

$$\int_C \frac{1}{z^2 - 8z + 1} dz$$

 (b) Use or not use part (a) to compute

$$\int_0^\pi \frac{1}{4 - \cos\theta} d\theta$$

Solution. The function

$$\frac{1}{z^2 - 8z + 1} = \frac{1}{(z - 4 - \sqrt{15})(z - 4 + \sqrt{15})}$$

has a singularity in $|z| < 1$ at $z = 4 - \sqrt{15}$. Therefore,

$$\begin{aligned}\int_C \frac{1}{z^2 - 8z + 1} dz &= 2\pi i \lim_{z = 4 - \sqrt{15}} \frac{1}{(z - 4 - \sqrt{15})(z - 4 + \sqrt{15})} \\ &= 2\pi i \left(\frac{1}{z - 4 - \sqrt{15}} \right) \bigg|_{z = 4 - \sqrt{15}} = -\frac{\pi i}{\sqrt{15}}\end{aligned}$$

That is,

$$\begin{aligned}\int_C \frac{1}{z^2 - 8z + 1} dz &= \int_{-\pi}^{\pi} \frac{de^{i\theta}}{e^{2i\theta} - 8e^{i\theta} + 1} \\ &= i \int_{-\pi}^{\pi} \frac{e^{i\theta}}{e^{2i\theta} - 8e^{i\theta} + 1} d\theta \\ &= i \int_{-\pi}^{\pi} \frac{1}{e^{i\theta} + e^{-i\theta} - 8} d\theta \\ &= -\frac{i}{2} \int_{-\pi}^{\pi} \frac{1}{4 - \cos\theta} d\theta \\ &= -i \int_0^{\pi} \frac{1}{4 - \cos\theta} d\theta\end{aligned}$$

Therefore,

$$\int_0^\pi \frac{1}{4 - \cos\theta} d\theta = \frac{\pi}{\sqrt{15}}$$

■

18. Compute the integral

$$\int_{-\pi}^{\pi} \frac{dx}{2 - (\cos x + \sin x)}.$$

Solution. Let $z = e^{ix}$. Then $dz = ie^{ix}dx$, $dx = -idz/z$ and hence

$$
\begin{aligned}
\int_{-\pi}^{\pi} \frac{dx}{2-(\cos x+\sin x)} &= \int_{-\pi}^{\pi} \frac{dx}{2-(e^{ix}+e^{-ix})/2-(e^{ix}-e^{-ix})/(2i)} \\
&= \int_{|z|=1} \frac{-idz}{2z-(z^2+1)/2-(z^2-1)/(2i)} \\
&= (i-1)\int_{|z|=1} \frac{dz}{z^2-2(1+i)z+i} \\
&= (i-1)\int_{|z|=1} \frac{dz}{(z-z_1)(z-z_2)} \\
&= \frac{2\pi i(i-1)}{z_2-z_1} = \sqrt{2}\pi
\end{aligned}
$$

where $z_1 = (1+\sqrt{2}/2)+(1+\sqrt{2}/2)i$ and $z_2 = (1-\sqrt{2}/2)+(1-\sqrt{2}/2)i$. ∎

19. Let $f(z)$ be an entire function satisfying

$$|f(z_1+z_2)| \le |f(z_1)|+|f(z_2)|$$

for all complex numbers z_1 and z_2. Show that $f(z)$ is a polynomial of degree at most 1.

Proof. We have

$$
\begin{aligned}
\sum_{k=1}^{n} f(z_k) &= f(z_1)+f(z_2)+\sum_{k=3}^{n} f(z_k) \\
&= f(z_1+z_2)+\sum_{k=3}^{n} f(z_k) \\
&= f(z_1+z_2)+f(z_3)+\sum_{k=4}^{n} f(z_k) \\
&= f(z_1+z_2+z_3)+\sum_{k=4}^{n} f(z_k) \\
&= \ldots = f(z_1+z_2+\ldots+z_n) = f\left(\sum_{k=1}^{n} x_k\right).
\end{aligned}
$$

Therefore,

$$\sum_{k=1}^{n} f(z_k) = f(z_1)+f(z_2)+\ldots+f(z_n) = f(z_1+z_2+\ldots+z_n) = f\left(\sum_{k=1}^{n} x_k\right)$$

for all complex numbers $z_1, z_2, \ldots, z_n$. Particularly, this holds for $z_1 = z_2 = \ldots = z_n = z/n$:

$$nf\left(\frac{z}{n}\right) = f(z)$$

for all $z \in \mathbb{C}$ and all positive integer n. Let M be the maximum of $|f(z)|$ for $|z| = 1$. Then

$$|f(z)| = n\left|f\left(\frac{z}{n}\right)\right| \le nM$$

for all z satisfying $|z| = n$.
By Cauchy Integral Formula,

$$f''(z_0) = \frac{1}{\pi i}\int_{|z|=n} \frac{f(z)}{(z-z_0)^3}dz$$

for $|z_0| < n$. Since

$$\left|\frac{f(z)}{(z-z_0)^3}\right| = \frac{|f(z)|}{|z-z_0|^3} \le \frac{nM}{(n-|z_0|)^3}$$

for $|z| = n$ and $|z_0| < n$,

$$\left|\frac{1}{\pi i}\int_{|z|=n} \frac{f(z)}{(z-z_0)^3}dz\right| \le \frac{2n^2M}{(n-|z_0|)^3}.$$

And since

$$\lim_{n\to\infty} \frac{2n^2M}{(n-|z_0|)^3} = \lim_{n\to\infty} \frac{2M/n}{(1-|z_0|/n)^3} = 0,$$

we conclude that $f''(z_0) = 0$ for all z_0. Therefore, $f'(z) \equiv a$ is a constant and $f(z) = az + b$ is a polynomial of degree at most 1. ∎

20. Let $f(z)$ be an entire function satisfying that $|f(z)| \le |z|^2$ for all z. Show that $f(z) \equiv az^2$ for some constant a satisfying $|a| \le 1$.

Proof. For every $z_0 \in \mathbb{C}$, we have

$$f'''(z_0) = \frac{3!}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^4}dz$$

for all $R > 0$. Since

$$\left|\frac{f(z)}{(z-z_0)^4}\right| \le \frac{|z|^2}{R^4} \le \frac{(R+|z_0|)^2}{R^4}$$

for $|z-z_0| = R$,

$$|f'''(z_0)| = \left|\frac{3!}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^4}dz\right| \le \frac{6(R+|z_0|)^2}{R^3}.$$

And since

$$\lim_{R\to\infty} \frac{6(R+|z_0|)^2}{R^3} = \lim_{R\to\infty} \frac{6}{R}\left(1+\frac{|z_0|}{R}\right)^2 = 0,$$

we conclude that $|f'''(z_0)| = 0$ and hence $f'''(z_0) = 0$ for every $z_0 \in \mathbb{C}$. Therefore, $f'''(z) \equiv 0$, $f''(z) \equiv 2a$, $f'(z) \equiv 2az + b$ and $f(z) \equiv az^2 + bz + c$ for some constants a, b and c.
Since $|f(z)| \le |z|^2$, $|az^2 + bz + c| \le |z|^2$ for all z. Take $z = 0$ and we obtain $|c| \le 0$. Hence $c = 0$. Therefore, $|az^2 + bz| \le |z|^2$ and hence $|az + b| \le |z|$ for all z. Take $z = 0$ again and we obtain $|b| \le 0$. Hence $b = 0$. So $|az^2| \le |z|^2$ and hence $|a| \le 1$. In conclusion, $f(z) = az^2$ with a satisfying $|a| \le 1$. ■

21. Let $f(z)$ be a complex polynomial of degree at least 2 and R be a positive number such that $f(z) \ne 0$ for all $|z| \ge R$. Show that

$$\int_{|z|=R} \frac{dz}{f(z)} = 0.$$

[Refer to: problem 5 in section 4.3]

Proof. Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $a_n \ne 0$ and $n = \deg f$. Since $f(z) \ne 0$ for $|z| \ge R$, $1/f(z)$ is analytic in $|z| \ge R$. Hence

$$\int_{|z|=R} \frac{dz}{f(z)} = \int_{|z|=r} \frac{dz}{f(z)}$$

for all $r \ge R$ by Cauchy Integral Theorem.
Since

$$|f(z)| \ge |a_n||z|^n - |a_{n-1}||z|^{n-1} - \cdots - |a_0|$$

we have

$$\left| \frac{1}{f(z)} \right| \le \frac{1}{|a_n|r^n - |a_{n-1}|r^{n-1} - \cdots - |a_0|}$$

for $|z| = r$ sufficiently large. It follows that

$$\left| \int_{|z|=r} \frac{dz}{f(z)} \right| \le \frac{2\pi r}{|a_n|r^n - |a_{n-1}|r^{n-1} - \cdots - |a_0|}$$

And since $n \ge 2$,

$$\begin{aligned} &\lim_{r\to\infty} \frac{2\pi r}{|a_n|r^n - |a_{n-1}|r^{n-1} - \cdots - |a_0|} \\ &= \lim_{r\to\infty} \frac{2\pi}{n|a_n|r^{n-1} - (n-1)|a_{n-1}|r^{n-1} - \cdots - |a_1|} = 0 \end{aligned}$$

by L'Hospital. Hence

$$\int_{|z|=R} \frac{dz}{f(z)} = \lim_{r\to\infty} \int_{|z|=r} \frac{dz}{f(z)} = 0.$$

■

3.3 Improper integrals

1. Compute the integral

$$\int_0^\infty \frac{xdx}{x^3+1}.$$

Solution. Consider the contour integral of $z/(z^3+1)$ along $L_R = [0,R]$, $C_R = \{z = Re^{it} : 0 \le t \le 2\pi/3\}$ and $M_R = \{te^{2\pi i/3} : 0 \le t \le R\}$. By Cauchy Integral Formula,

$$\int_{L_R} \frac{zdz}{z^3+1} + \int_{C_R} \frac{zdz}{z^3+1} - \int_{M_R} \frac{zdz}{z^3+1} = \int_{|z-e^{\pi i/3}|=1/2} \frac{zdz}{z^3+1}$$

By Cauchy Integral Formula,

$$\begin{aligned}\int_{|z-e^{\pi i/3}|=1/2} \frac{zdz}{z^3+1} &= \frac{2\pi i \exp(\pi i/3)}{(\exp(\pi i/3)+1)(\exp(\pi i/3)-\exp(-\pi i/3))} \\ &= \frac{2\pi \exp(\pi i/3)}{(\exp(\pi i/3)+1)\sqrt{3}}.\end{aligned}$$

For z lying on C_R,

$$\left|\frac{z}{z^3+1}\right| \le \frac{R}{R^3-1}$$

and hence

$$\left|\int_{C_R} \frac{zdz}{z^3+1}\right| \le \frac{2\pi R}{3(R^3-1)}$$

It follows that

$$\lim_{R\to\infty} \int_{C_R} \frac{zdz}{z^3+1} = 0$$

And

$$\int_{M_R} \frac{zdz}{z^3+1} = \exp(4\pi i/3) \int_0^R \frac{xdx}{x^3+1}$$

Therefore, we have

$$(1-\exp(4\pi i/3)) \int_0^\infty \frac{xdx}{x^3+1} = \frac{2\pi \exp(\pi i/3)}{(\exp(\pi i/3)+1)\sqrt{3}}$$

and hence

$$\int_0^\infty \frac{xdx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

■

2. Compute the integral

$$\int_0^\infty \frac{\cos x}{x^4+1}dx.$$

Solution. Since $\cos x/(x^4+1)$ is even,

$$\int_0^\infty \frac{\cos x}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{x^4+1}dx.$$

Actually, we have

$$\int_0^\infty \frac{\cos x}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{ix}}{x^4+1}dx$$

since $e^{ix} = \cos x + i\sin x$.
Consider the contour integral of $e^{iz}/(z^4+1)$ along the path $L_R = [-R, R]$ and $C_R = \{|z| = R, \mathrm{Im}(z) \geq 0\}$, oriented counterclockwise. By Cauchy Integral Theorem, we have

$$\begin{aligned}&\int_{L_R} \frac{e^{iz}}{z^4+1}dz + \int_{C_R} \frac{e^{iz}}{z^4+1}dz \\ = &\int_{|z-e^{\pi i/4}|=1/2} \frac{e^{iz}}{z^4+1}dz + \int_{|z-e^{3\pi i/4}|=1/2} \frac{e^{iz}}{z^4+1}dz\end{aligned}$$

By Cauchy Integral Formula,

$$\begin{aligned}\int_{|z-e^{\pi i/4}|=1/2} \frac{e^{iz}}{z^4+1}dz &= \frac{2\pi i e^{i(\sqrt{2}+i\sqrt{2})/2}}{(e^{\pi i/4}-e^{3\pi i/4})(e^{\pi i/4}-e^{-\pi i/4})(e^{\pi i/4}-e^{-3\pi i/4})} \\ &= \frac{\pi(1-i)\exp((-\sqrt{2}+i\sqrt{2})/2)}{2\sqrt{2}}\end{aligned}$$

and similarly,

$$\int_{|z-e^{3\pi i/4}|=1/2} \frac{e^{iz}}{z^4+1}dz = \frac{\pi(1+i)\exp((-\sqrt{2}-i\sqrt{2})/2)}{2\sqrt{2}}$$

Therefore,

$$\int_{L_R} \frac{e^{iz}}{z^4+1}dz + \int_{C_R} \frac{e^{iz}}{z^4+1}dz = \frac{\pi e^{-\sqrt{2}/2}}{\sqrt{2}}(\cos(\sqrt{2}/2) + \sin(\sqrt{2}/2)).$$

For z lying on C_R, $y = \mathrm{Im}(z) \geq 0$ and hence $|e^{iz}| = e^{-y} \leq 1$. Hence

$$\left|\frac{e^{iz}}{z^4+1}\right| \leq \frac{1}{R^4-1}$$

and it follows that

$$\left|\int_{C_R} \frac{e^{iz}}{z^4+1}dz\right| \leq \frac{\pi R}{R^4-1}$$

Since

$$\lim_{R\to\infty} \frac{\pi R}{R^4 - 1} = 0,$$

we conclude that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^4+1} dz = 0.$$

Therefore,

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{x^4+1} dx &= \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix}}{x^4+1} dx \\
&= \frac{1}{2} \lim_{R\to\infty} \int_{L_R} \frac{e^{iz}}{z^4+1} dz \\
&= \frac{\pi e^{-\sqrt{2}/2}}{2\sqrt{2}} \left(\cos\left(\frac{\sqrt{2}}{2}\right) + \sin\left(\frac{\sqrt{2}}{2}\right) \right).
\end{aligned}$$

■

4. Series

4.1 Taylor and Laurent series

1. Find the Taylor series of the following functions and their radii of convergence:

   (a) $z\sinh(z^2)$ at $z = 0$;

   (b) e^z at $z = 2$;

   (c) $\dfrac{z^2+z}{(1-z)^2}$ at $z = -1$.

Solution. (a) Since $e^z = \sum_{n=0}^{\infty} z^n/n!$,

$$
\begin{aligned}
z\sinh(z^2) &= z\left(\frac{e^{z^2}-e^{-z^2}}{2}\right)\\
&= \frac{z}{2}\left(\sum_{n=0}^{\infty}\frac{z^{2n}}{n!} - \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{n!}\right)\\
&= \sum_{n=0}^{\infty}\frac{1-(-1)^n}{2}\frac{z^{2n+1}}{n!}\\
&= \sum_{m=0}^{\infty}\frac{z^{4m+3}}{(2m+1)!}
\end{aligned}
$$

where we observe that $(1-(-1)^n)/2 = 0$ if $n = 2m$ is even and 1 if $n = 2m+1$ is odd. Since $f(z)$ is entire, the radius of convergence is ∞.

(b) Let $w = z - 2$. Then $z = w + 2$ and

$$e^z = e^{w+2} = e^2 e^w = e^2 \sum_{n=0}^{\infty} \frac{w^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2 (z-2)^n}{n!}.$$

Since $f(z)$ is entire, the radius of convergence is ∞.

(c) Let $w = z + 1$. Then

$$\frac{z^2+z}{(1-z)^2} = \frac{w^2-w}{(2-w)^2} = 1 - \frac{3}{2-w} + \frac{2}{(2-w)^2}.$$

We have

$$-\frac{3}{2-w} = -\frac{3}{2}\frac{1}{1-(w/2)} = -\frac{3}{2}\sum_{n=0}^{\infty}\frac{w^n}{2^n} = -\sum_{n=0}^{\infty}\frac{3w^n}{2^{n+1}}$$

and

$$\begin{aligned}
\frac{2}{(2-w)^2} &= \left(\frac{2}{2-w}\right)' = \left(\frac{1}{1-(w/2)}\right)' \\
&= \left(\sum_{n=0}^{\infty}\frac{w^n}{2^n}\right)' = \sum_{n=0}^{\infty}\frac{nw^{n-1}}{2^n} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)w^n}{2^{n+1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{z^2+z}{(1-z)^2} &= 1 - \sum_{n=0}^{\infty}\frac{3w^n}{2^{n+1}} + \sum_{n=0}^{\infty}\frac{(n+1)w^n}{2^{n+1}}. \\
&= \sum_{n=1}^{\infty}\frac{(n-2)(z+1)^n}{2^{n+1}}.
\end{aligned}$$

Since $f(z)$ is analytic in $|z+1| < 2$ and has a singularity at $z = 1$, the radius of convergence is 2.

■

2. Find the Taylor series of $(\cos z)^2$ at $z = \pi$.

Solution. Let $w = z - \pi$. Then

$$\begin{aligned}
(\cos z)^2 &= (\cos(z+\pi))^2 = (\cos w)^2 \\
&= \left(\frac{e^{iw}+e^{-iw}}{2}\right)^2 = \frac{1}{4}(e^{2iw} + e^{-2iw} + 2) \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(2i)^n w^n}{n!} + \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-2i)^n w^n}{n!} + \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(2i)^{2n} w^{2n}}{(2n)!} = \frac{1}{2} + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n-1} w^{2n}}{(2n)!} \\
&= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1} w^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}(z-\pi)^{2n}}{(2n)!}
\end{aligned}$$

■

3. Let $f(z)$ be a function analytic at 0 and $g(z) = f(z^2)$. Show that $g^{(2n-1)}(0) = 0$ for all positive integers n.

Proof. Since $f(z)$ is analytic at 0, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in some disk $|z| < r$. Therefore, $g(z) = f(z^2) = \sum_{n=0}^{\infty} a_n z^{2n}$ in $|z| < \sqrt{r}$ and hence

$$\sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} z^m = \sum_{n=0}^{\infty} a_n z^{2n}$$

And since the power series representation of an analytic function is unique, we must have $g^{(m)}(0) = 0$ for m is odd, i.e., $m = 2n-1$ for all positive integers n. ■

4. Find a power-series expansion of the function $f(z) = \dfrac{1}{3-z}$ about the point 4i, and calculate the radius of convergence.

Solution. Notice that

$$\begin{aligned}
\frac{1}{3-z} &= \frac{1}{(3-4iz)-(z-4i)} \\
&= \frac{1}{3-4i} \cdot \frac{1}{1 - \dfrac{z-4i}{3-4i}} \\
&= \frac{1}{3-4i} \sum_{n=0}^{\infty} \left(\frac{z-4i}{3-4i} \right)^n \quad \text{for} \quad \left| \frac{z-4i}{3-4i} \right| < 1
\end{aligned}$$

That is, for $|z-4i| < |3-4i| < 5$. Thus

$$\frac{1}{3-z} = \sum_{n=0}^{\infty} \frac{(z-4i)^n}{(3-4i)^{n+1}}$$

with radius of convergence 5. ■

5. Find a Laurent-series expansion of the function $f(z) = z^{-1} \sinh(z^{-1})$ about the point 0, and classify the singularity at 0.

Solution. For $g(z) = \sinh z$ we know that

$$g^{(n)}(z) = \begin{cases} \sinh z, & \text{when } n \text{ is even;} \\ \cosh z, & \text{when } n \text{ is odd} \end{cases}$$

Thus

$$g^{(n)}(0) = \begin{cases} \sinh(0) = 0, & \text{when } n \text{ is even;} \\ \cosh(0) = 1, & \text{when } n \text{ is odd} \end{cases}$$

In this case, the Maclaurin series for $g(z) = \sinh z$ is:

$$z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

The Laurent series for $g(z^{-1}) = \sinh z^{-1}$ is:

$$\frac{1}{z} + \frac{1}{3!z^3} + \frac{1}{5!z^5} + \cdots$$

Thus Laurent series for $f(z) = z^{-1}g(z^{-1}) = z^{-1}\sinh z^{-1}$ is

$$\frac{1}{z^2} + \frac{1}{3!z^4} + \frac{1}{5!z^6} + \cdots$$

Notice that $f(z) = z^{-1}\sinh z^{-1}$ is analytic for $z \neq 0$, which means that $z = 0$ is an isolated singularity and is, in fact, an essential singularity.

■

6. Consider the function
$$f(z) = \frac{\sin z}{\cos(z^3) - 1}.$$
Classify the singularity at $z = 0$ and calculate the residue.

Solution. Notice that f has an isolated singularity at $z = 0$. Thus, expanding numerator and denominator in Taylor series we have

$$\begin{aligned}
f(z) &= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}{-\frac{z^6}{2!} + \frac{z^{12}}{4!} - \frac{z^{18}}{6!}\cdots} \\
&= \frac{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right)}{-\frac{z^6}{2!}\left(1 - \frac{2z^6}{4!} + \frac{2z^{12}}{6!} - \cdots\right)} \\
&= \frac{-2}{z^5} \cdot \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}{1 - \frac{2z^6}{4!} + \frac{2z^{12}}{6!} - \cdots} \\
&= \frac{-2}{z^5}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right) \cdot \frac{1}{1 - \frac{2z^6}{4!} + \frac{2z^{12}}{6!} - \cdots}
\end{aligned}$$

Let

$$g(z) = \frac{2}{4!} - \frac{2z^6}{6!} + \frac{2z^{12}}{8!} - \cdots$$

Thus we have

$$f(z) = \frac{-2}{z^5}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right) \cdot \frac{1}{1 - z^6 g(z)} \tag{4.1}$$

where $g(z)$ is analytic and nonzero at $z = 0$, in fact, $g(0) = 1/12$. So for ε sufficiently small, if $|z| < \varepsilon$, then $|g(z)| < 1$. Thus for $|z| < \min\{\varepsilon, 1\}$, we can expand

$$\frac{1}{1 - z^6 g(z)}$$

in a geometric series. Hence

$$\begin{aligned} f(z) &= \frac{-2}{z^5}\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\cdots\right)\left(1+z^6g(z)+z^{12}\left(g(z)\right)^2+\cdots\right) \\ &= \left(\frac{-2}{z^5}+\frac{2}{3!z^3}-\frac{2}{5!z}+\frac{2z}{7!}+\cdots\right)\left(1+z^6g(z)+z^{12}\left(g(z)\right)^2+\cdots\right) \end{aligned}$$

So the residue of f at $z=0$ is $\dfrac{-2}{5!}=\dfrac{-1}{60}$. ■

7. Prove that the coefficients c_n in the expansion

$$\frac{1}{1-z-z^2}=\sum_{n=0}^{\infty} c_n z^n$$

satisfy the recurrence relation $c_0=c_1=1$, $c_n=c_{n-1}+c_{n-2}$ for $n\geq 2$. What is the radius of convergence of the series? What would be a good name for the c_n's?

Solution. Let $S=\sum\limits_{n=0}^{\infty} c_n z^n$ for z such that the series converges.

Thus

$$zS=\sum_{n=0}^{\infty} c_n z^{n+1}=\sum_{n=1}^{\infty} c_{n-1}z^n$$

and

$$z^2S=\sum_{n=0}^{\infty} c_n z^{n+2}=\sum_{n=2}^{\infty} c_{n-2}z^n$$

Then we have

$$\begin{aligned} S-zS-z^2S &= \sum_{n=0}^{\infty} c_n z^n-\sum_{n=1}^{\infty} c_{n-1}z^n-\sum_{n=2}^{\infty} c_{n-2}z^n \\ &= c_0+c_1z+\sum_{n=2}^{\infty} c_n z^n-c_0z-\sum_{n=2}^{\infty} c_{n-1}z^n-\sum_{n=2}^{\infty} c_{n-2}z^n \\ &= c_0+(c_1-c_0)z+\sum_{n=2}^{\infty}(c_n-c_{n-1}-c_{n-2})z^n \end{aligned}$$

Now, we know that

$$\frac{1}{1-z-z^2}=\sum_{n=0}^{\infty} c_n z^n=S$$

then

$$S-zS-z^2S = 1$$

Thus

$$c_0=1,\quad c_1-c_0=0,\quad \text{and}\quad c_n-c_{n-1}-c_{n-2}=0\quad \text{for}\quad n\geq 2.$$

Therefore $c_0=c_1=1$, and $c_n=c_{n-1}+c_{n-2}$ for $n\geq 2$. ■

8. Find the Laurent series of the function

$$f(z) = \frac{z+4}{z^2(z^2+3z+2)}$$

in

(a) $0 < |z| < 1$;
(b) $1 < |z| < 2$;
(c) $|z| > 2$;
(d) $0 < |z+1| < 1$.

Solution. We write $f(z)$ as a sum of partial fractions:

$$\frac{z+4}{z^2(z^2+3z+2)} = -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z+1} - \frac{1}{2(z+2)}.$$

For $0 < |z| < 1$,

$$\frac{3}{z+1} = 3\sum_{n=0}^{\infty}(-1)^n z^n$$

and

$$-\frac{1}{2(z+2)} = -\frac{1}{4}\frac{1}{1+(z/2)} = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n}.$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{5}{2z} + \frac{2}{z^2} + 3\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n} \\ &= \frac{2}{z^2} - \frac{5}{2z} + \sum_{n=0}^{\infty}(-1)^n(3-2^{-n-2})z^n \end{aligned}$$

For $1 < |z| < 2$,

$$\frac{3}{z+1} = \frac{3}{z}\frac{1}{1+(1/z)} = \frac{3}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$$

and

$$-\frac{1}{2(z+2)} = -\frac{1}{4}\frac{1}{1+(z/2)} = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n}.$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^n} \\ &= \sum_{n=2}^{\infty}\frac{3(-1)^n}{z^{n+1}} - \frac{1}{z^2} + \frac{1}{2z} - \sum_{n=0}^{\infty}(-1)^n 2^{-n-2} z^n. \end{aligned}$$

For $2 < |z| < \infty$,

$$\frac{3}{z+1} = \frac{3}{z}\frac{1}{1+(1/z)} = \frac{3}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$$

and

$$-\frac{1}{2(z+2)} = -\frac{1}{2z}\frac{1}{1+(2/z)} = -\frac{1}{2z}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^n}.$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \frac{1}{2z}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^n} \\ &= \sum_{n=3}^{\infty}\frac{(-1)^{n+1}(3-2^{n-2})}{z^n}. \end{aligned}$$

For $0 < |z+1| < 1$, we let $w = z+1$ and then

$$\frac{z+4}{z^2(z^2+3z+2)} = \frac{5}{2(1-w)} + \frac{2}{(1-w)^2} + \frac{3}{w} - \frac{1}{2(w+1)}.$$

For $0 < |w| < 1$,

$$\frac{5}{2(1-w)} = \frac{5}{2}\sum_{n=0}^{\infty} w^n,$$

$$\frac{2}{(1-w)^2} = \left(\frac{2}{1-w}\right)' = \left(\sum_{n=0}^{\infty} 2w^n\right) = \sum_{n=0}^{\infty} 2(n+1)w^n$$

and

$$-\frac{1}{2(w+1)} = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n w^n.$$

Therefore,

$$\begin{aligned} f(z) &= \frac{3}{w} + \sum_{n=0}^{\infty}\left(2n + \frac{9}{2} - \frac{(-1)^n}{2}\right)w^n \\ &= \frac{3}{z+1} + \sum_{n=0}^{\infty}\left(2n + \frac{9}{2} - \frac{(-1)^n}{2}\right)(z+1)^n. \end{aligned}$$

■

9. Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains and specify these domains.

Solution. Since $f(z)$ is analytic at $z \neq 0, \pm i$, it is analytic in $0 < |z| < 1$ and $1 < |z| < \infty$.
For $0 < |z| < 1$,

$$\begin{aligned} f(z) &= \frac{1}{z(1+z^2)} = \frac{1}{z}\left(\frac{1}{1-(-z^2)}\right) \\ &= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n} = \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \end{aligned}$$

and for $1 < |z| < \infty$,

$$\begin{aligned} f(z) &= \frac{1}{z(1+z^2)} = \frac{1}{z^3}\left(\frac{1}{1-(-z^{-2})}\right) \\ &= \frac{1}{z^3}\sum_{n=0}^{\infty}(-1)^n z^{-2n} = \sum_{n=0}^{\infty}(-1)^n z^{-2n-3} \end{aligned}$$

■

10. Let

$$f(z) = \frac{z^2}{z^2-3z+2}$$

Find the Laurent series of $f(z)$ in each of the following domains:

(a) $1 < |z| < 2$
(b) $1 < |z-3| < 2$

Solution. First, we write $f(z)$ as a sum of partial fractions:

$$\frac{z^2}{z^2-3z+2} = 1 + \frac{3z-2}{(z-2)(z-1)} = 1 + \frac{4}{z-2} - \frac{1}{z-1}$$

In $1 < |z| < 2$,

$$\begin{aligned} \frac{z^2}{z^2-3z+2} &= 1 - \frac{2}{1-z/2} - \frac{1}{z}\frac{1}{1-1/z} \\ &= 1 - 2\sum_{n=0}^{\infty} 2^{-n} z^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} \\ &= -1 - \sum_{n=1}^{\infty} 2^{1-n} z^n - \sum_{n=1}^{\infty} z^{-n} \end{aligned}$$

In $1 < |z-3| < 2$,

$$\begin{aligned}
\frac{z^2}{z^2-3z+2} &= 1 + \frac{4}{(z-3)+1} - \frac{1}{2+(z-3)} \\
&= 1 + \frac{4}{z-3}\left(\frac{1}{1+1/(z-3)}\right) - \frac{1}{2}\left(\frac{1}{1+(z-3)/2}\right) \\
&= 1 + \frac{4}{z-3}\sum_{n=0}^{\infty}(-1)^n(z-3)^{-n} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n 2^{-n}(z-3)^n \\
&= \frac{1}{2} + 4\sum_{n=1}^{\infty}(-1)^{n+1}(z-3)^{-n} - \sum_{n=1}^{\infty}(-1)^n 2^{-n-1}(z-3)^n
\end{aligned}$$

■

11. Let

$$f(z) = \frac{z^2}{z^2-z-2}$$

Find the Laurent series of $f(z)$ in each of the following domains:

(a) $1 < |z| < 2$
(b) $0 < |z-2| < 1$

Solution. First, we write $f(z)$ as a sum of partial fractions:

$$\frac{z^2}{z^2-z-2} = 1 + \frac{z+2}{(z-2)(z+1)} = 1 + \frac{4}{3(z-2)} - \frac{1}{3(z+1)}$$

In $1 < |z| < 2$,

$$\begin{aligned}
\frac{z^2}{z^2-z-2} &= 1 - \frac{2}{3}\frac{1}{1-z/2} - \frac{1}{3z}\frac{1}{1+1/z} \\
&= 1 - \frac{2}{3}\sum_{n=0}^{\infty}2^{-n}z^n - \frac{1}{3z}\sum_{n=0}^{\infty}(-1)^n z^{-n} \\
&= \frac{1}{3} - \frac{1}{3}\sum_{n=1}^{\infty}2^{1-n}z^n + \frac{1}{3}\sum_{n=1}^{\infty}(-1)^n z^{-n}
\end{aligned}$$

In $0 < |z-2| < 1$,

$$\begin{aligned}
\frac{z^2}{z^2-z-2} &= 1 + \frac{4}{3(z-2)} - \frac{1}{3}\frac{1}{3+(z-2)} \\
&= 1 + \frac{4}{3(z-2)} - \frac{1}{9}\frac{1}{1+(z-2)/3} \\
&= 1 + \frac{4}{3(z-2)} - \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n 3^{-n}(z-2)^n \\
&= \frac{8}{9} + \frac{4}{3(z-2)} + \sum_{n=1}^{\infty}(-1)^{n+1}3^{-n-2}(z-2)^n
\end{aligned}$$

■

12. Find the Laurent series of

$$\frac{1}{e^{z^2}-1}$$

in z up to z^6 and show the series converges in $0<|z|<\sqrt{2\pi}$.

Solution. Let $f(z)=1/(e^z-1)$. Since

$$e^z-1=\sum_{n=0}^{\infty}\frac{z^n}{n!}-1=\sum_{n=1}^{\infty}\frac{z^n}{n!}=z\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}$$

e^z-1 has a zero at 0 of multiplicity one and hence $f(z)$ has pole at 0 of order 1. So the Laurent series of $f(z)$ is given by

$$f(z)=\sum_{n=-1}^{\infty}a_nz^n=\frac{a_{-1}}{z}+a_0+a_1z+a_2z^2+a_3z^3+\sum_{n\ge 4}a_nz^n$$

in $0<|z|<r$ for some $r>0$.
Since $(e^z-1)f(z)=1$, we have

$$1=\left(a_{-1}+a_0z+a_1z^2+a_2z^3+a_3z^4+\sum_{n\ge 5}a_{n-1}z^n\right)\left(1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\frac{z^4}{120}+\sum_{n\ge 5}\frac{z^n}{(n+1)!}\right).$$

Comparing the coefficients of 1, z, z^2, z^3 and z^4 on both sides, we obtain

$$\begin{cases} a_{-1}=1\\ a_0+\dfrac{a_{-1}}{2}=0\\ a_1+\dfrac{a_0}{2}+\dfrac{a_{-1}}{6}=0\\ a_2+\dfrac{a_1}{2}+\dfrac{a_0}{6}+\dfrac{a_{-1}}{24}=0\\ a_3+\dfrac{a_2}{2}+\dfrac{a_1}{6}+\dfrac{a_0}{24}+\dfrac{a_{-1}}{120}=0 \end{cases}$$

Solving it, we have $a_{-1}=1$, $a_0=-1/2$, $a_1=1/12$, $a_2=0$ and $a_3=-1/720$. Hence

$$f(z)=\frac{1}{z}-\frac{1}{2}+\frac{z}{12}-\frac{z^3}{720}+\sum_{n\ge 4}a_nz^n$$

and

$$\frac{1}{e^{z^2}-1}=f(z^2)=\frac{1}{z^2}-\frac{1}{2}+\frac{z^2}{12}-\frac{z^6}{720}+\sum_{n\ge 4}a_nz^{2n}.$$

Note that $f(z)$ is analytic in $\{z:e^z-1\ne 0\}=\{z\ne 2n\pi i\}$. So it is analytic in $0<|z|<2\pi$. Therefore, $f(z^2)$ is analytic in $0<|z^2|<2\pi$, i.e., $0<|z|<\sqrt{2\pi}$. So the series converges in $0<|z|<\sqrt{2\pi}$. ∎

4.2 Classification of singularities

1. For each of the following complex functions, do the following:
   - find all its singularities in $\mathbb{C}$;
   - write the principal part of the function at each singularity;
   - for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
   - compute the residue of the function at each singularity.

   (a) $f(z) = \dfrac{1}{(\cos z)^2}$

   Solution. $f(z)$ is singular at $\cos z = 0$, i.e., $z = n\pi + \pi/2$. Let $w = z - n\pi - \pi/2$. Then

$$\frac{1}{(\cos z)^2} = \frac{1}{(\cos(w + n\pi + \pi/2))^2} = \frac{1}{(\sin w)^2}.$$

   Since $\sin w$ has a zero of multiplicity one at $w = 0$, $f(z)$ has a pole of order 2 at $z = n\pi + \pi/2$. So

$$\frac{1}{(\sin w)^2} = \frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + \sum_{n \geq 0} a_n w^n.$$

   Since

$$(\sin w)^2 = \left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} \right)^2 = w^2 + \sum_{n=4}^{\infty} b_n w^n$$

   we have

$$1 = \left(\frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + \sum_{n \geq 0} a_n w^n \right) \left(w^2 + \sum_{n=4}^{\infty} b_n w^n \right).$$

   Comparing the coefficients of 1 and w on both sides, we obtain $a_{-2} = 1$ and $a_{-1} = 0$. So the principal part of $f(z)$ at $z = n\pi + \pi/2$ is

$$\frac{1}{(z - n\pi - \pi/2)^2}$$

   with residue 0. ■

(b) $f(z) = (1-z^3)\exp\left(\frac{1}{z}\right)$

Solution. Since $e^z = \sum_{n=0}^{\infty} z^n/n!$,

$$
\begin{aligned}
(1-z^3)\exp\left(\frac{1}{z}\right) &= (1-z^3)\sum_{n=0}^{\infty}\frac{1}{n!z^n} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!z^n} - \sum_{n=0}^{\infty}\frac{1}{n!z^{n-3}} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!z^n} - \sum_{n=4}^{\infty}\frac{1}{n!z^{n-3}} - \sum_{n=0}^{3}\frac{z^{3-n}}{n!} \\
&= 1 + \sum_{n=1}^{\infty}\frac{1}{n!z^n} - \sum_{n=1}^{\infty}\frac{1}{(n+3)!z^n} - \sum_{n=0}^{3}\frac{z^{3-n}}{n!} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n+3)!}\right)\frac{1}{z^n} + 1 - \sum_{n=0}^{3}\frac{z^{3-n}}{n!}
\end{aligned}
$$

Therefore, $f(z)$ has an essential singularity at $z=0$ with principal part

$$
\sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n+3)!}\right)\frac{1}{z^n}
$$

and residue

$$
\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!} - \frac{1}{4!} = \frac{23}{24}.
$$

■

(c) $f(z) = \frac{\sin z}{z^{2010}}$

Solution. Since

$$
\begin{aligned}
\frac{\sin z}{z^{2010}} &= \frac{1}{z^{2010}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n-2009}}{(2n+1)!} \\
&= \sum_{n=0}^{1004}\frac{(-1)^n z^{2n-2009}}{(2n+1)!} + \sum_{n=1005}^{\infty}\frac{(-1)^n z^{2n-2009}}{(2n+1)!}
\end{aligned}
$$

$f(z)$ has a pole of order 2009 at $z=0$ with principal part

$$
\sum_{n=0}^{1004}\frac{(-1)^n z^{2n-2009}}{(2n+1)!}
$$

and with residue

$$
\operatorname*{Res}_{z=0}\frac{\sin z}{z^{2010}} = \frac{(-1)^{1004}}{(2\cdot 1004+1)!} = \frac{1}{2009!}.
$$

■

(d) $f(z) = \dfrac{e^z}{1-z^2}$

Solution. Since $1-z^2 = (1-z)(1+z)$, $f(z)$ has poles of order 1 at 1 and -1. Therefore,

$$\operatorname*{Res}_{z=1} \frac{e^z}{1-z^2} = \left.\frac{e^z}{(1-z^2)'}\right|_{z=1} = -\frac{e}{2}$$

and

$$\operatorname*{Res}_{z=-1} \frac{e^z}{1-z^2} = \left.\frac{e^z}{(1-z^2)'}\right|_{z=-1} = \frac{1}{2e}.$$

And the principal parts of $f(z)$ at $z=1$ and $z=-1$ are

$$-\frac{e}{2(z-1)} \text{ and } \frac{1}{2e(z+1)}$$

respectively.

■

(e) $f(z) = (1-z^2)\exp\left(\dfrac{1}{z}\right)$

Solution. The function has a singularity at 0 where

$$\begin{aligned}
(1-z^2)\exp\left(\frac{1}{z}\right) &= (1-z^2)\sum_{n=0}^{\infty}\frac{1}{(n!)z^n} \\
&= \sum_{n=0}^{\infty}\frac{1}{(n!)z^n} - \sum_{n=0}^{\infty}\frac{1}{(n!)z^{n-2}} \\
&= 1 + \sum_{n=1}^{\infty}\frac{1}{(n!)z^n} - \sum_{n=3}^{\infty}\frac{1}{(n!)z^{n-2}} - (z^2+z+\frac{1}{2}) \\
&= -z^2 - z + \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{(n!)z^n} - \sum_{n=1}^{\infty}\frac{1}{(n+2)!z^n} \\
&= -z^2 - z + \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n+2)!}\right)z^{-n}
\end{aligned}$$

So the principal part is

$$\sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n+2)!}\right)z^{-n}$$

the function has an essential singularity at 0 and

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!} - \frac{1}{3!} = \frac{5}{6}$$

■

(f) $f(z) = \dfrac{1}{(\sin z)^2}$

Solution. The function has singularities at $k\pi$ for $k \in \mathbb{Z}$. At $z = k\pi$, we let $w = z - k\pi$ and then

$$\begin{aligned}
\frac{1}{(\sin z)^2} = \frac{1}{(\sin w)^2} &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!}\right)^{-1} \\
&= \frac{1}{w^2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n+1)!}\right)^{-1} \\
&= \frac{1}{w^2}\left(1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!}\right)^{-1} \\
&= \frac{1}{w^2}\sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!}\right)^{m} \\
&= \frac{1}{w^2}\left(1 + \sum_{n=2}^{\infty} a_n w^n\right)
\end{aligned}$$

So the principal part at $k\pi$ is

$$\frac{1}{(z-k\pi)^2}$$

the function has a pole of order 2 at $k\pi$ and

$$\operatorname*{Res}_{z=k\pi} f(z) = 0$$

■

(g) $f(z) = \dfrac{1-\cos z}{z^2}$

Solution. The function has a singularity at 0 where

$$\begin{aligned}
\frac{1-\cos z}{z^2} &= \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right) = \frac{1}{z^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n-2}}{(2n)!}
\end{aligned}$$

So the principal part is 0, the function has a removable singularity at 0 and

$$\operatorname*{Res}_{z=0} f(z) = 0$$

■

(h) $f(z) = \dfrac{e^z}{z(z-1)^2}$

Solution. The function has two singularities at 0 and 1. At $z = 0$,

$$\begin{aligned}\frac{e^z}{z(z-1)^2} &= \frac{1}{z}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}(n+1)z^n\right)\\ &= \frac{1}{z}\left(1+\sum_{n=1}^{\infty}a_n z^n\right)\end{aligned}$$

So the principal part at 0 is $1/z$, the function has a pole of order 1 at 0 and

$$\operatorname*{Res}_{z=0} f(z) = 1$$

At $z = 1$, we let $w = z - 1$ and then

$$\begin{aligned}\frac{e^z}{z(z-1)^2} = \frac{e^{1+w}}{(1+w)w^2} &= \frac{e}{w^2}\left(\sum_{n=0}^{\infty}\frac{w^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n w^n\right)\\ &= \frac{e}{w^2}\left(1+w+\sum_{n=2}^{\infty}\frac{w^n}{n!}\right)\left(1-w+\sum_{n=2}^{\infty}(-1)^n w^n\right)\\ &= \frac{e}{w^2}\left(1+\sum_{n=2}^{\infty}a_n w^n\right)\end{aligned}$$

So the principal part at 1 is

$$\frac{e}{(z-1)^2}$$

the function has a pole of order 2 at 1 and

$$\operatorname*{Res}_{z=1} f(z) = 0$$

■

(i) $f(z) = \tan z$

Solution. The function has singularities at $\{\cos z = 0\} = \{z = k\pi + \pi/2 : k \in \mathbb{Z}$. At $z = k\pi + \pi/2$, we let $w = z - k\pi - \pi/2$ and then

$$\tan z = \tan\left(w + k\pi + \frac{\pi}{2}\right) = \tan\left(w + \frac{\pi}{2}\right) = -\frac{\cos w}{\sin w}$$

Since $\sin w$ has a zero of multiplicity 1 at $w = 0$, $\tan z$ has a pole of order 1 at $z = k\pi + \pi/2$. Therefore

$$\operatorname*{Res}_{z=k\pi+\pi/2} \tan z = \operatorname*{Res}_{w=0}\left(-\frac{\cos w}{\sin w}\right) = -\left.\frac{\cos w}{(\sin w)'}\right|_{w=0} = -1$$

and the principal part of $\tan z$ at $z = k\pi + \pi/2$ is

$$-\frac{1}{w} = -\frac{1}{z - k\pi - \pi/2}.$$

■

(j) $f(z) = (1-z^2)\sin\left(\frac{1}{z}\right)$

Solution. The function has a singularity at 0 where

$$\begin{aligned}
(1-z^2)\sin\left(\frac{1}{z}\right) &= (1-z^2)\sum_{n=0}^{\infty}\frac{(-1)^n}{((2n+1)!)z^{2n+1}}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{((2n+1)!)z^{2n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{((2n+1)!)z^{2n-1}}\\
&= -z + \sum_{n=0}^{\infty}\frac{(-1)^n}{((2n+1)!)z^{2n+1}} - \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!)z^{2n-1}}\\
&= -z + \sum_{n=0}^{\infty}\frac{(-1)^n}{((2n+1)!)z^{2n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{((2n+3)!)z^{2n+1}}\\
&= -z + \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(2n+1)!}+\frac{1}{(2n+3)!}\right)z^{-2n-1}
\end{aligned}$$

So the principal part is

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(2n+1)!}+\frac{1}{(2n+3)!}\right)z^{-2n-1}.$$

Therefore, the function has an essential singularity at 0 and

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!}+\frac{1}{3!} = \frac{7}{6}.$$

■

(k) $f(z) = \frac{e^z}{z^{2011}}$

Solution. The function has a singularity at 0 where

$$\begin{aligned}
\frac{e^z}{z^{2011}} &= \frac{1}{z^{2011}}\sum_{n=0}^{\infty}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{z^{n-2011}}{n!} = \sum_{n=0}^{2010}\frac{z^{n-2011}}{n!} + \sum_{n=2011}^{\infty}\frac{z^{n-2011}}{n!}
\end{aligned}$$

Therefore, the principal part of $f(z)$ at $z=0$ is

$$\sum_{n=0}^{2010}\frac{z^{n-2011}}{n!}$$

and $f(z)$ has a pole of order 2011 and residue

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2010!}$$

at $z=0$. ■

(1) $f(z) = \dfrac{\cos z}{z^2 - z^3}$

Solution. The function has singularities at $\{z^2 - z^3 = 0\} = \{z = 0, 1\}$. At $z = 0$, $z^2 - z^3$ has a zero of multiplicity 2 and hence $f(z)$ has a pole of order 2. Suppose that the Laurent series of $f(z)$ at $z = 0$ is given by

$$\frac{\cos z}{z^2 - z^3} = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n.$$

Hence

$$(z^2 - z^3)\left(\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n\right) = \cos z = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

Comparing the coefficients of 1 and z on both sides, we obtain that $a_{-2} = 1$ and $a_{-1} - a_{-2} = 0$ and hence $a_{-1} = a_{-2} = 1$. So the principal part of $f(z)$ at $z = 0$ is

$$\frac{1}{z^2} + \frac{1}{z}$$

with residue

$$\operatorname*{Res}_{z=0} f(z) = 1.$$

At $z = 1$, $z^2 - z^3$ has a zero of multiplicity 1 and hence $f(z)$ has a pole of order 1. Hence

$$\operatorname*{Res}_{z=1} \frac{\cos z}{z^2 - z^3} = \left.\frac{\cos z}{(z^2 - z^3)'}\right|_{z=1} = -\cos(1)$$

and the principal part of $f(z)$ at $z = 1$ is

$$-\frac{\cos(1)}{z-1}.$$

■

4.3 Applications of residues

1. Calculate

$$\int_C \frac{8-z}{z(4-z)} dz,$$

where C is the circle of radius 7, centre 0, negatively oriented.

Solution. Observe that

$$f(z) = \frac{8-z}{z(4-z)} = \frac{8-2z+z}{z(4-z)} = \frac{2}{z} + \frac{1}{(4-z)} = \frac{2}{z} - \frac{1}{(z-4)}$$

The function f has two singularities on $\mathbb{C}$, $z=0$ and $z=4$. Both are inside C. At $z=4$, $-1/(z-4)$ is analytic and then

$$\operatorname*{Res}_{z=0} f(z) = 2$$

Similarly, since $2/z$ is analytic at $z=4$,

$$\operatorname*{Res}_{z=4} f(z) = -1$$

From Cauchy's residue theorem, and considering that C is negatively oriented, we have that

$$\begin{aligned}\int_C \frac{8-z}{z(4-z)}dz &= -2\pi i\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=4} f(z)\right)\\ &= -2\pi i(2-1) = -2\pi i\end{aligned}$$

■

2. Compute the integral

$$\int_0^\pi \frac{d\theta}{2-\cos\theta}.$$

Solution. Since $1/(2-\cos\theta)$ is even,

$$\int_0^\pi \frac{d\theta}{2-\cos\theta} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{2-\cos\theta}.$$

Let $z=e^{i\theta}$. Then $\cos\theta = (z+z^{-1})/2$ and $d\theta = -iz^{-1}dz$. Hence

$$\begin{aligned}\int_0^\pi \frac{d\theta}{2-\cos\theta} &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{2-\cos\theta}\\ &= \int_{|z|=1}\frac{-idz}{2z(2-(z+z^{-1})/2)}\\ &= i\int_{|z|=1}\frac{dz}{z^2-4z+1}.\end{aligned}$$

The function

$$\frac{1}{z^2-4z+1} = \frac{1}{(z-2-\sqrt{3})(z-2+\sqrt{3})}$$

has a singularity in $|z|<1$ at $z=2-\sqrt{3}$. Therefore,

$$\begin{aligned}\int_{|z|=1}\frac{dz}{z^2-4z+1} &= 2\pi i \operatorname*{Res}_{z=2-\sqrt{3}}\frac{1}{z^2-4z+1}\\ &= 2\pi i \left.\frac{1}{(z^2-4z+1)'}\right|_{z=2-\sqrt{3}} = -\frac{\pi i}{\sqrt{3}}.\end{aligned}$$

Therefore,

$$\int_0^\pi \frac{d\theta}{2-\cos\theta} = \frac{\pi}{\sqrt{3}}$$

■

3. Let $a, b \in \mathbb{R}$ such that $a^2 > b^2$. Calculate the integral

$$\int_0^\pi \frac{d\theta}{a + b\cos\theta}.$$

Answer: $\dfrac{\pi}{\sqrt{a^2 - b^2}}$

4. Evaluate the contour integral of the following functions around the circle $|z| = 2011$ oriented counterclockwise:

(a) $\dfrac{1}{\sin z}$;

(b) $\dfrac{1}{e^{2z} - e^z}$.

Solution. (a) $f(z) = 1/\sin z$ is analytic in $\{z \neq n\pi : n \in \mathbb{Z}\}$. It has a pole of order one at $n\pi$ since $(\sin z)'|_{z=n\pi} = \cos(n\pi) = (-1)^n \neq 0$. So

$$\operatorname*{Res}_{z=n\pi} \frac{1}{\sin z} = \frac{1}{\cos(n\pi)} = (-1)^n.$$

Therefore,

$$\begin{aligned}\int_{|z|=2011} \frac{dz}{\sin z} &= 2\pi i \sum_{|n\pi|<2011} \operatorname*{Res}_{z=n\pi} \frac{1}{\sin z} \\ &= 2\pi i \sum_{|n|\leq 640} (-1)^n = 2\pi i.\end{aligned}$$

(b) $f(z) = 1/(e^{2z} - e^z)$ is analytic in

$$\{e^{2z} - e^z \neq 0\} = \{e^z \neq 1\} = \{z \neq 2n\pi i : n \in \mathbb{Z}\}.$$

Since $(e^{2z} - e^z)'|_{z=2n\pi i} = 1 \neq 0$, $f(z)$ has a pole of order one at $2n\pi i$. So

$$\operatorname*{Res}_{z=2n\pi i} \frac{1}{e^{2z} - e^z} = \left.\frac{1}{2e^{2z} - e^z}\right|_{z=2n\pi i} = 1.$$

Therefore,

$$\begin{aligned}\int_{|z|=2011} \frac{dz}{e^{2z} - e^z} &= 2\pi i \sum_{|2n\pi i|<2011} \operatorname*{Res}_{z=2n\pi i} \frac{1}{e^{2z} - e^z} \\ &= 2\pi i \sum_{|2n\pi i|<2011} 1 = 2\pi i \sum_{|n|\leq 320} 1 = 1282\pi i.\end{aligned}$$

■

5. Let
$$f(z) = (z-a_1)(z-a_2)\ldots(z-a_n)$$
be a complex polynomial with $n \geq 2$ distinct roots $a_1, a_2, \ldots, a_n$.
   (a) Prove that
$$\int_{|z|=R} \frac{dz}{f(z)} = 2\pi i \sum_{k=1}^{n} \frac{1}{\prod_{j\neq k}(a_k - a_j)}$$
   for $R > |a_k|$ ($k = 1, 2, \ldots, n$).
   (b) Use (a) and Cauchy Integral Theorem to prove that
$$\sum_{k=1}^{n} \frac{1}{\prod_{j\neq k}(a_k - a_j)} = 0$$
   for all distinct complex numbers $a_1, a_2, \ldots, a_n$.

Proof. By Residue theorem,
$$\int_{|z|=R} \frac{dz}{f(z)} = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} \frac{1}{f(z)}.$$
At each a_k, $1/f(z)$ has a pole of order one and
$$\operatorname*{Res}_{z=a_k} \frac{1}{f(z)} = \operatorname*{Res}_{z=a_k} \frac{\left(\prod_{j\neq k}(z-a_j)\right)^{-1}}{z-a_k} = \frac{1}{\prod_{j\neq k}(a_k - a_j)}.$$
Therefore,
$$\int_{|z|=R} \frac{dz}{f(z)} = 2\pi i \sum_{k=1}^{n} \frac{1}{\prod_{j\neq k}(a_k - a_j)}.$$
Since $\deg(f(z)) = n \geq 2$,
$$\int_{|z|=R} \frac{dz}{f(z)} = 0$$
by Problem 21 in Section 3.2. Therefore,
$$\sum_{k=1}^{n} \frac{1}{\prod_{j\neq k}(a_k - a_j)} = 0.$$
■

6. Use Cauchy Integral Theorem or Residue Theorem to show that
$$\frac{1}{2\pi i} \int_{C_N} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}$$
and conclude that
$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Solution. The function $f(z) = 1/(z^2 \sin z)$ has singularities at $z = n\pi$ for $n \in \mathbb{Z}$. So

$$\frac{1}{2\pi i}\int_{C_N} \frac{dz}{z^2 \sin z} = \sum_{n=-N}^{n} \operatorname*{Res}_{z=n\pi} \frac{1}{z^2 \sin z}$$

by Residue Theorem.
At $z = n\pi$ for a nonzero integer n,

$$z^2\Big|_{n\pi} \neq 0 \quad \text{and} \quad (\sin z)'\Big|_{n\pi} \neq 0.$$

Therefore, $1/(z^2 \sin z)$ has a pole of order 1 at $n\pi$ for $n \neq 0$. It follows that

$$\operatorname*{Res}_{z=n\pi} \frac{1}{z^2 \sin z} = \frac{1}{z^2 (\sin z)'}\Big|_{z=n\pi} = \frac{1}{n^2\pi^2 \cos(n\pi)} = \frac{(-1)^n}{n^2\pi^2}$$

for $n \neq 0$.
At $z = 0$, $z^2 \sin z$ has a zero multiplicity 3 and hence $1/(z^2 \sin z)$ has a pole of order 3. Suppose that the Laurent series of $f(z)$ at $z = 0$ is given by

$$\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n\geq 0} a_n z^n.$$

Then

$$\begin{aligned} & z^2 \sin z \left(\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n\geq 0} a_n z^n \right) \\ = & \left(1 - \frac{z^2}{3!} + \sum_{n\geq 3} b_n z^n \right) \left(a_{-3} + a_{-2}z + a_{-1}z^2 + \sum_{n\geq 3} a_n z^n \right) = 1. \end{aligned}$$

Comparing the coefficients of 1, z and z^2 on both sides, we have

$$\begin{cases} a_{-3} = 1 \\ a_{-2} = 0 \\ a_{-1} - \dfrac{a_{-3}}{6} = 0 \end{cases}$$

Solving the equation, we obtain $a_{-1} = 1/6$, $a_{-2} = 0$ and $a_{-3} = 1$. So

$$\operatorname*{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{1}{6}.$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_N} \frac{dz}{z^2 \sin z} &= \sum_{n=-N}^{n} \operatorname*{Res}_{z=n\pi} \frac{1}{z^2 \sin z} \\ &= \frac{1}{6} + \sum_{n=-N}^{n=-1} \frac{(-1)^n}{n^2\pi^2} + \sum_{n=1}^{n=N} \frac{(-1)^n}{n^2\pi^2}. \end{aligned}$$

We observe that

$$\frac{(-1)^n}{n^2\pi^2} = \frac{(-1)^{-n}}{(-n)^2\pi^2}$$

and hence we obtain

$$\frac{1}{2\pi i}\int_{C_N} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}.$$

By Problem 14 in section 3.2, we have

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{C_N} \frac{dz}{z^2 \sin z} = 0$$

Consequently,

$$\frac{1}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} = 0$$

That is,

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

■

4.3.1 Improper integrals

1. Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4+x^2+1} dx$

Solution. since $e^{ix} = \cos x + i \sin x$,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+x^2+1} dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^4+x^2+1} dx\right)$$

Actually, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+x^2+1} dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4+x^2+1} dx$$

in this case since $\sin x/(x^4+x^2+1)$ is odd.
Consider the contour integral of $e^{iz}/(z^4+z^2+1)$ along the path $L_R = [-R, R]$ and $C_R = \{|z| = R, \operatorname{Im}(z) \geq 0\}$, oriented counterclockwise. By CIT or residue theorem, we have

$$\int_{L_R} \frac{e^{iz}}{z^4+z^2+1} dz + \int_{C_R} \frac{e^{iz}}{z^4+z^2+1} dz$$
$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} \frac{e^{iz}}{z^4+z^2+1}$$

where $z_1, z_2, \ldots, z_n$ are the singularities of $e^{iz}/(z^4+z^2+1)$ inside the region $\{|z| < R, \mathrm{Im}(z) > 0\}$.
We find the singularities of $e^{iz}/(z^4+z^2+1)$ by solving $z^4+z^2+1=0$: we observe that $(z^2-1)(z^4+z^2+1) = z^6-1$. So the function has four singularities $\pm e^{\pi i/3}$ and $\pm e^{2\pi i/3}$. Two of them $e^{\pi i/3}$ and $e^{2\pi i/3}$ lie above the real axis. Therefore,

$$\begin{aligned}&\int_{L_R} \frac{e^{iz}}{z^4+z^2+1}dz + \int_{C_R} \frac{e^{iz}}{z^4+z^2+1}dz \\ &= 2\pi i \left(\operatorname*{Res}_{z=e^{\pi i/3}} \frac{e^{iz}}{z^4+z^2+1} + \operatorname*{Res}_{z=e^{2\pi i/3}} \frac{e^{iz}}{z^4+z^2+1} \right).\end{aligned}$$

Since all zeros of z^4+z^2+1 have multiplicity one, all poles of

$$e^{iz}/(z^4+z^2+1)$$

have order one. Therefore,

$$\operatorname*{Res}_{z=e^{\pi i/3}} \frac{e^{iz}}{z^4+z^2+1} = \left.\frac{e^{iz}}{(z^4+z^2+1)'}\right|_{z=e^{\pi i/3}} = \frac{\exp((-\sqrt{3}+i)/2)}{\sqrt{3}i-3}$$

and

$$\operatorname*{Res}_{z=e^{2\pi i/3}} \frac{e^{iz}}{z^4+z^2+1} = \left.\frac{e^{iz}}{(z^4+z^2+1)'}\right|_{z=e^{2\pi i/3}} = \frac{\exp((-\sqrt{3}-i)/2)}{\sqrt{3}i+3}.$$

Hence

$$\int_{L_R} \frac{e^{iz}}{z^4+z^2+1}dz + \int_{C_R} \frac{e^{iz}}{z^4+z^2+1}dz = \frac{\pi}{3}\left(\sqrt{3}\cos\left(\frac{1}{2}\right) + 3\sin\left(\frac{1}{2}\right)\right).$$

For z lying on C_R, $y = \mathrm{Im}(z) \geq 0$ and hence $|e^{iz}| = e^{-y} \leq 1$. Hence

$$\left|\frac{e^{iz}}{z^4+z^2+1}\right| \leq \frac{1}{R^4-R^2-1}$$

and it follows that

$$\left|\int_{C_R} \frac{e^{iz}}{z^4+z^2+1}dz\right| \leq \frac{\pi R}{R^4-R^2-1}$$

Since

$$\lim_{R\to\infty} \frac{\pi R}{R^4-R^2-1} = 0,$$

we conclude that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^4+z^2+1}dz = 0.$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{x^4+x^2+1} dx &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4+x^2+1} dx \\ &= \lim_{R\to\infty} \int_{L_R} \frac{e^{iz}}{z^4+z^2+1} dz \\ &= \frac{\pi}{3}\left(\sqrt{3}\cos\left(\frac{1}{2}\right)+3\sin\left(\frac{1}{2}\right)\right).\end{aligned}$$

■

2. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2} dx.$$

Solution. since $e^{ix} = \cos x + i\sin x$,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x^2+2} dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+2x^2+2} dx\right)$$

Consider the contour integral of $e^{iz}/(z^2+2z^2+2)$ along the path $L_R = [-R,R]$ and $C_R = \{|z| = R, \operatorname{Im}(z) \geq 0\}$, oriented counterclockwise.
Since $e^{iz}/(z^2+2z+2)$ has two isolated singularities at $-1 \pm i$ with $-1+i$ lying inside the curve $L_R \cup C_R$, we have

$$\begin{aligned}&\int_{L_R} \frac{e^{iz}}{z^2+2z^2+2} dz + \int_{C_R} \frac{e^{iz}}{z^2+2z^2+2} dz \\ &= 2\pi i \operatorname*{Res}_{z=-1+i} \frac{e^{iz}}{z^2+2z+2} \\ &= 2\pi i \left.\frac{e^{iz}}{(z^2+2z+2)'}\right|_{z=-1+i} \\ &= \frac{2\pi i \exp(-i-1)}{2i} = \frac{\pi}{e}(\cos(1) - i\sin(1))\end{aligned}$$

by Cauchy Integral Theorem or residue theorem.
For z lying on C_R, $y = \operatorname{Im}(z) \geq 0$ and hence $|e^{iz}| = e^{-y} \leq 1$. Hence

$$\left|\frac{e^{iz}}{z^2+2z^2+2}\right| \leq \frac{1}{R^2-2R^2-2}$$

and it follows that

$$\left|\int_{C_R} \frac{e^{iz}}{z^2+2z+2} dz\right| \leq \frac{\pi R}{R^2-2R^2-2}$$

Since

$$\lim_{R\to\infty} \frac{\pi R}{R^2-2R^2-2} = 0,$$

we conclude that

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z^2+2z^2+2}dz=0.$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\sin x}{x^2+2x^2+2}dx &= \mathrm{Im}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+2x^2+2}dx\right)\\ &= \mathrm{Im}\left(\lim_{R\to\infty}\int_{L_R}\frac{e^{iz}}{z^2+2z^2+2}dz\right)\\ &= -\frac{\pi}{e}\sin(1).\end{aligned}$$

■

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