الفيزياء الرياضية -١-

Mathematical Physics -1-

PHYS 201

First Term 2020-2021

Series of Applications in Physics

Application II: Matrices

Introdcution

Applications of matrices are faced in most scientific fields. Particularly, in **physics**, including classical mechanics, optics, electromagnetism, quantum mechanics, and quantum electrodynamics, they are used to study physical phenomena, such as the motion of rigid bodies.

Coordinate Transformation

In many problems we will need to use different coordinate systems in order to describe different vector physical quantities. The above operations, written in component form, only make sense once all the vectors involved are described with respect to the same frame.

In several situations we need to know how these physical quantities are given in other frames.

We need hence to see how the components of a vector are transformed when we change the reference frame.

→ We focus here on the cases dealing with rotation

Application: Coordinate Transformation- Rotation

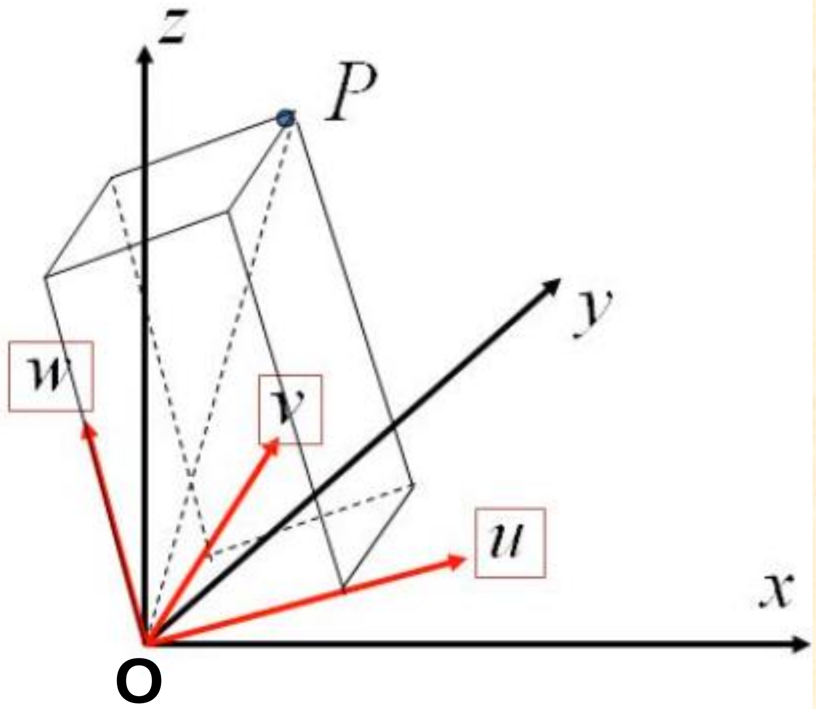
A vector **OP** given in two different frames

- Coordinate Transformation

$$\vec{P}_{xyz} = p_x \mathrm{i}_x + p_y \mathrm{j}_y + p_z \mathrm{k}_z$$

$$\vec{P}_{uvw} = p_u \mathrm{i}_u + p_v \mathrm{j}_v + p_w \mathrm{k}_w$$

$$P_{xyz} = RP_{uvw}$$



How to relate the coordinate in these two frames?

Application: Coordinate Transformation- Rotation

- Basic Rotation
  - p_x, p_y and p_z represent the projections of P onto OX, OY, OZ axes, respectively
  - Since $P = p_u \mathrm{i}_\mathrm{u} + p_v \mathrm{j}_\mathrm{v} + p_w \mathrm{k}_\mathrm{w}$

$$\boxed{p_x} = \mathrm{i}_\mathrm{x} \cdot P = \mathrm{i}_\mathrm{x} \cdot \mathrm{i}_\mathrm{u} p_u + \mathrm{i}_\mathrm{x} \cdot \mathrm{j}_\mathrm{v} p_v + \mathrm{i}_\mathrm{x} \cdot \mathrm{k}_\mathrm{w} p_w$$

$$\boxed{p_y} = \mathrm{j}_\mathrm{y} \cdot P = \mathrm{j}_\mathrm{y} \cdot \mathrm{i}_\mathrm{u} p_u + \mathrm{j}_\mathrm{y} \cdot \mathrm{j}_\mathrm{v} p_v + \mathrm{j}_\mathrm{y} \cdot \mathrm{k}_\mathrm{w} p_w$$

$$\boxed{p_z} = \mathrm{k}_\mathrm{z} \cdot P = \mathrm{k}_\mathrm{z} \cdot \mathrm{i}_\mathrm{u} p_u + \mathrm{k}_\mathrm{z} \cdot \mathrm{j}_\mathrm{v} p_v + \mathrm{k}_\mathrm{z} \cdot \mathrm{k}_\mathrm{w} p_w$$

Coordinates in the Frame (x,y,z)

Application: Coordinate Transformation- Rotation

The previous 3 relations (in 3 dimensions) can be written in matrix-form as function of the scalar product between the unit vectors of the two systems of coordinates

- Basic Rotation Matrix

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathrm{i_x} \cdot \mathrm{i_u} & \mathrm{i_x} \cdot \mathrm{j_v} & \mathrm{i_x} \cdot \mathrm{k_w} \\ \mathrm{j_y} \cdot \mathrm{i_u} & \mathrm{j_y} \cdot \mathrm{j_v} & \mathrm{j_y} \cdot \mathrm{k_w} \\ \mathrm{k_z} \cdot \mathrm{i_u} & \mathrm{k_z} \cdot \mathrm{j_v} & \mathrm{k_z} \cdot \mathrm{k_w} \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

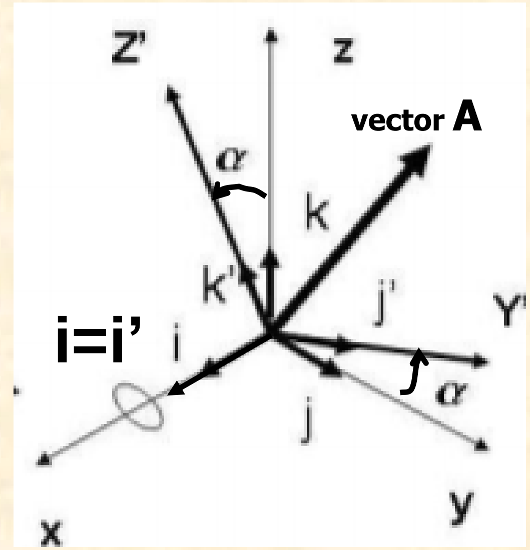
Application: Rotation about X-axis

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} \boldsymbol{i}_1' \cdot \boldsymbol{i}_1 & \boldsymbol{i}_1' \cdot \boldsymbol{i}_2 & \boldsymbol{i}_1' \cdot \boldsymbol{i}_3 \\ \boldsymbol{i}_2' \cdot \boldsymbol{i}_1 & \boldsymbol{i}_2' \cdot \boldsymbol{i}_2 & \boldsymbol{i}_2' \cdot \boldsymbol{i}_3 \\ \boldsymbol{i}_3' \cdot \boldsymbol{i}_1 & \boldsymbol{i}_3' \cdot \boldsymbol{i}_2 & \boldsymbol{i}_3' \cdot \boldsymbol{i}_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}.$$

Called: transformation matrix
مصفوفة الانتقال
(ضرب بين متجهات الوحدة)

الاحداثيات في المرجع ٢

الاحداثيات في المرجع ١



Simply, becomes

Because unit vectors
Have magnitude 1

$$\mathbf{A}' = \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} \cos(\theta_{11}) & \cos(\theta_{12}) & \cos(\theta_{13}) \\ \cos(\theta_{21}) & \cos(\theta_{22}) & \cos(\theta_{23}) \\ \cos(\theta_{31}) & \cos(\theta_{32}) & \cos(\theta_{33}) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}.$$

Property: $\mathbf{A} = [T]^T \mathbf{A}'.$ the transpose of the Transformation Matrix will transform A' back to

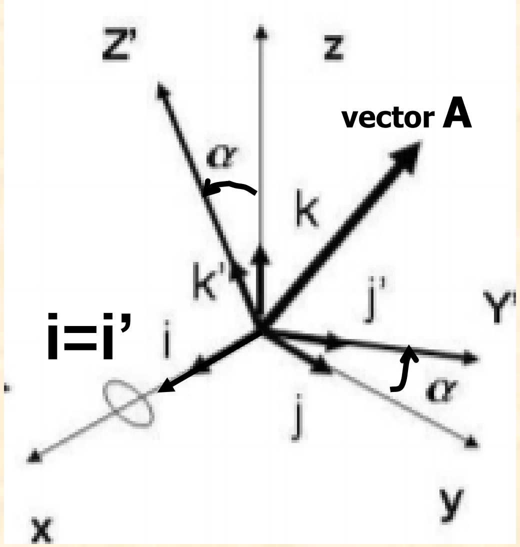
Application: Rotation about X-axis

$$R_x(\alpha) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}}$$

transformation matrix

مصفوفة الانتقال

(دوران حول محور X)



Ex:

A point $\boldsymbol{P}$ is represented in the (x_1, x_2, x_3) coordinate system as $\boldsymbol{P(3,1,2)}$. In another coordinate system, the same point is represented as $P(x_1', x_2', x_3')$, where $\boldsymbol{x_2}$ has been rotated toward $\boldsymbol{x_3}$ around the $\boldsymbol{x_1}$***-axis*** by an angle of $\boldsymbol{\pi/4}$.

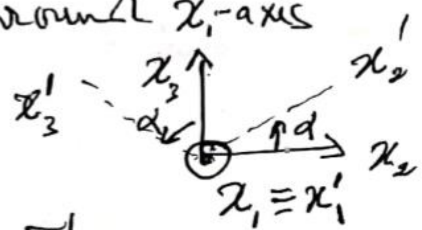
a- find the rotation matrix.

b- determine the new coordinates $P(x_1', x_2', x_3')$

<u>Application:</u> Rotation about X-axis

<u>Sol:</u>

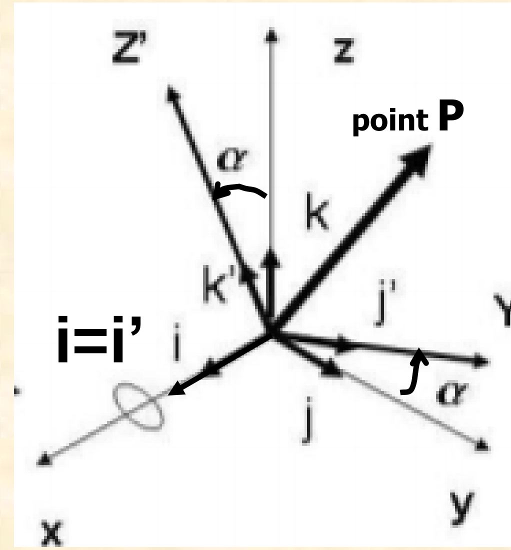
a/ Rotation Matrix around x_1-axis counterclockwise



$$M_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \qquad \alpha = \pi/4$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

b/ $$P\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$



<u>Note</u>: If you compute the magnitude of the OP vector in the two systems Of coordinates you will find them equal.

Any scalar quantity is invariant under coordinates transformation

(و هي خاصية مهمة في الفيزياء)