Indian J. Pure Appl. Math., **44**(3): , June 2013

ON THE REPRESENTATION OF BIHARMONIC FUNCTIONS WITH SINGULARITIES IN $\mathbb{R}^n$

M.A. Al-Gwaiz* and V. Anandam**

**Department of Mathematics, King Saud University,* 700 108 *India*
P.O. Box 2455, *Riyadh* 11451, *Saudi Arabia*
***The Institute of Mathematical Sciences, C.I.T. Campus,*
Taramani, Chennai 600 113, *India*
e-mails: malgwaiz@ksu.edu.sa, vanandam@hotmail.com

(*Received* 29 *May* 2012; *accepted* 10 *October* 2012)

Biharmonic functions are defined on Euclidean spaces, Riemannian manifolds, infinite trees, and more generally on abstract harmonic spaces. In this note, we consider biharmonic functions b defined on annular sets $\Omega \setminus K$ and obtain Laurent-type decompositions for b in the Euclidean spaces and in infinite trees. Particular importance is given to the investigation when b extends as a distribution on Ω.

Key words : Biharmonic distributions; Laurent decomposition; infinite trees.

1. INTRODUCTION

Consider a distribution S with support in an interval A. Let θ be a test function in $\mathscr{D}(\mathbb{R})$ equal to 1 on A. Then it is known (see for example, Hervé [7, [Proposition

3.22]) that the map T defined as $T(\varphi) = S(\theta\varphi)$ for every $\varphi \in C^\infty(\mathbb{R})$ is a linear map on $C^\infty(\mathbb{R})$ such that $T(\varphi) = S(\varphi)$ if $\varphi \in \mathscr{D}(\mathbb{R})$; there exists a bounded interval I, an integer m and positive coefficients $c_0, c_1, \ldots, c_m$ such that $|T\varphi| \leq \sum_{k=0}^{m} c_k \sup_I |\varphi^k|$ for any $\varphi \in C^\infty(\mathbb{R})$. This result leads to the question: if K is a compact subset of an open set Ω in $\mathbb{R}^n$, and if S is a distribution in $\Omega \setminus K$, then, does there exist a distribution T on Ω such that $T = S$ on $\Omega \setminus K$? We answer this question in the case when S is a regular distribution in $\Omega \setminus K$ defined by a biharmonic function b on $\Omega \setminus K$, that is, b is a C^∞ function on $\Omega \setminus K$ such that $\Delta^2 b = 0$.

2. Preliminaries

Let Δ be the Laplacian operator in $\mathbb{R}^n$, $n \geq 2$. The fundamental singularity S_n of the operator $\Delta^2 = \Delta\Delta$ at $x = 0$ satisfies the differential equation $\Delta^2 S_n = \delta$ in $\mathbb{R}^n$ in the sense of distributions, where δ is the Dirac measure supported at the origin. As a function of $r = |x|$, S_n can be constructed by solving the differential equation

$$\Delta S_n(r) = \frac{1}{r^{n-1}} \left(r^{n-1} S_n'(r)\right)' = E_n(r), \quad r > 0,$$

E_n being the fundamental singularity of Δ at $x = 0$, which is given by

$$E_n(r) = \begin{cases} \dfrac{1}{2\pi} \log r, & n = 2 \\ \dfrac{-1}{(n-2)\sigma_n r^{n-2}}, & n \geq 3. \end{cases}$$

This yields

$$S_n(r) = \begin{cases} \dfrac{1}{8\pi} r^2 (\log r - 1), & n = 2 \\ -\dfrac{1}{8\pi} r, & n = 3 \\ -\dfrac{1}{8\pi^2} \log r, & n = 4 \\ \dfrac{1}{2(n-2)(n-4)} \dfrac{1}{r^{n-4}}, & n \geq 5. \end{cases}$$

Here σ_n is the surface area of the unit sphere in $\mathbb{R}^n$. Note that S_n extends as a continuous function in $\mathbb{R}^n$ when $n = 2$ and $n = 3$, but not when $n \geq 4$. If Ω is an open set in $\mathbb{R}^n$ and K is a compact subset of Ω, we shall define $H_0(\mathbb{R}^n \backslash K)$ to be the set of harmonic functions in $\mathbb{R}^n \backslash K$ which behave like E_n as $|x| \to \infty$. More precisely, $s \in H_0(\mathbb{R}^n \backslash K)$ if s is harmonic in $\mathbb{R}^n \backslash K$ and

$$s(x) = \alpha E_n(|x|) + o\left(1/\left|x\right|^{n-2}\right) \quad \text{as } |x| \to \infty,$$

for some constant α. In [3], the following two results were proved.

***Theorem* 2.1** — *If u is a harmonic function in $\Omega \backslash K$, it can be represented uniquely as a sum $u = s + t$ of a function $s \in H_0(\mathbb{R}^n \backslash K)$ and a harmonic function t in Ω such that $s(x) - \alpha \log |x| \to 0$ as $|x| \to \infty$ in $\mathbb{R}^2$ and $|s(x)| \leq p(x)$ outside a compact set in $\mathbb{R}^n,\ n \geq 3$, where $p(x)$ is a potential in $\mathbb{R}^n$.*

***Theorem* 2.2** — *If u is a biharmonic function in $\Omega \backslash K$, it can be represented as a sum $u = p + q$ of a function p, which is biharmonic in $\mathbb{R}^n \backslash K$ with $\Delta p \in H_0(\mathbb{R}^n \backslash K)$, and a biharmonic function q in Ω. This representation of u is unique up to an additive harmonic function in $\mathbb{R}^n$.*

3. Biharmonic Functions in the Neighbourhood of a Singular Point

***Theorem* 3.1** — *Suppose Ω is an open set in $\mathbb{R}^n$, $x_0 \in \Omega$. Let u be a biharmonic function in $\Omega \backslash \{x_0\}$. Then there is a decomposition of u in the form $u = p + q$ on $\Omega \setminus \{x_0\}$, where q is biharmonic on Ω and $p(x) = \alpha S_n(|x - x_0|) + g(x)$. Here α is a uniquely determined constant and $g(x)$ is biharmonic on $\mathbb{R}^n \setminus \{x_0\}$, such that $\Delta g(x) - \beta \log |x| \to 0$ when $|x| \to \infty$ in $\mathbb{R}^2$ for some β, and $|\Delta g(x)| \leq s(x)$ outside a compact set for some potential s on $\mathbb{R}^n$ if $n \geq 3$. This decomposition of u is unique up to an additive harmonic function in $\mathbb{R}^n$.*

PROOF : By Theorem 2.2, $u = p + q$ in $\Omega \setminus \{x_0\}$, where p is biharmonic in $\mathbb{R}^n \setminus \{x_0\}$ and q is biharmonic in Ω. Since Δp is harmonic in $\mathbb{R}^n \setminus \{x_0\}$, it can be written as $\Delta p(x) = \alpha E_n(x) + h(x)$ where $h(x)$ is harmonic in $\mathbb{R}^n \setminus \{x_0\}$ and tends to 0 as $|x| \to \infty$ if $n \geq 3$. Then $p(x) = \alpha S_n(|x - x_0|) + g(x)$ where $g(x)$

is biharmonic and $\Delta g(x) = h(x)$ such that $h(x) - \beta \log |x| \to 0$ when $|x| \to \infty$ in $\mathbb{R}^2$ and $|h(x)| \leq s(x)$ where $s(x)$ is a potential in $\mathbb{R}^n$, if $n \geq 3$.

Suppose $u = p_1 + q_1$ is another such decomposition with $p_1(x) = \alpha_1 S_n(|x - x_0|) + g_1(x)$. Then

$$B = \begin{cases} p - p_1, & \text{in } \mathbb{R}^n \setminus \{x_0\}; \\ q - q_1, & \text{in } \Omega. \end{cases}$$

is biharmonic on $\mathbb{R}^n$ and $\Delta B = (\alpha - \alpha_1) E_n(|x - x_0|) + [h(x) - h_1(x)]$ in $\mathbb{R}^n \setminus \{x_0\}$. In $\mathbb{R}^2$, since $[h(x) - h_1(x)]$ is bounded outside a compact set, its flux at infinity is 0; since ΔB is harmonic in $\mathbb{R}^2$, its flux at infinity is also 0. Consequently, $\alpha = \alpha_1$. The conditions on p and p_1 imply that $\Delta B \to 0$ when $|x| \to \infty$ and $n = 2$; and $|\Delta B(x)|$ is dominated by a potential on $\mathbb{R}^n$ outside a compact set if $n \geq 3$. In either case, ΔB being harmonic on $\mathbb{R}^n$, we conclude that $\Delta B = 0$, that is B is harmonic on $\mathbb{R}^n$. ∎

In the above sum $\alpha S_n(|x - x_0|) + g(x)$ representing $p(x)$ in Ω, it is worth investigating the conditions under which the function g can be left out; for then u becomes (up to a multiplicative constant) a fundamental solution for the operator Δ^2. The answer is provided by Bôcher's theorem (see [6]), which states that any positive harmonic function in the punctured unit disc $\{x \in \mathbb{R}^n : 0 < |x| < 1\}$ can be represented as a sum $cE_n(|x|) + h(x)$, where c is a constant and h is a harmonic function in the unit disc $\{x \in \mathbb{R}^n : |x| < 1\}$. Clearly, the same conclusion holds if the harmonic function is merely bounded on one side. Thus, with $\Delta u(x) = \alpha E_n(|x - x_0|) + \Delta g(x) + \Delta q(x)$ in $\Omega \setminus \{x_0\}$, we immediately conclude that if Δu is bounded on one side in a neighbourhood of x_0, then so is $\alpha E_n(|x - x_0|) + \Delta g(x)$ and hence Δg has the form $\beta E_n(|x - x_0|) + H(x)$ where $H(x)$ is harmonic in Ω. Consequently g may be dropped from the representation of u. Thus, we have

***Theorem* 3.2** — *Let Ω be an open set in $\mathbb{R}^n,\ x_0 \in \Omega$. Let u be a biharmonic function in $\Omega \setminus \{x_0\}$. Then the following statements are equivalent:*

(i) $u(x) = \alpha S_n(|x - x_0|) + q(x)$ in $\Omega \setminus \{x_0\}$ and q is biharmonic in Ω.
(ii) $\Delta^2 u = \alpha \delta_{x_0}$ in Ω.

(iii) u is biharmonic in $\Omega\backslash\{x_0\}$ and Δu is bounded on one side in a neighbourhood of x_0.

That settles the case of the removable singularity of g at x_0. The next question we can ask is: what other types of singularity can g have at x_0? This question will now be addressed.

4. Isolated Singularity of a Biharmonic Function

In this section we consider the question: let u be biharmonic in $\Omega \setminus \{x_0\}$. Is there a distribution T on Ω such that T restricted to $\Omega \setminus \{x_0\}$ is defined by u? Following the standard notation from calculus in $\mathbb{R}^n$, we take $x = (x_1, ..., x_n)$ to be a point in $\mathbb{R}^n$ and $\partial_k = \partial/\partial x_k,\ 1 \leq k \leq n$. If $\beta = (\beta_1, ..., \beta_n)$ is an n-tuple of nonnegative integers, then ∂^β denotes the differential operator

$$\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} = \frac{\partial^{|\beta|}}{\partial_1^{\beta_1} x_1 \cdots \partial_n^{\beta_n} x_n}$$

of order $|\beta| = \beta_1 + ... + \beta_n$.

If u is a biharmonic function in $\Omega\backslash\{x_0\}$ which extends to a distribution in Ω, then the distribution $\Delta^2 u$ is supported at the single point x_0. Such a distribution, according to a standard result of the theory (see [9] or [1, Theorem 3.2], for example), can only be a finite linear combination of the Dirac measure at x_0 and its derivatives. That is, there is a nonnegative integer m and constants c_β such that

$$\Delta^2 u = \sum_{|\beta| \leq m} c_\beta \partial^\beta \delta_{x_0}.$$

The general solution of this equation is

$$\begin{aligned} u(x) &= \left(\sum\nolimits_{|\beta|\leq m} c_\beta \partial^\beta \delta_{x_0}\right) * S_n\left(|x|\right) + q(x) \\ &= \sum_{|\beta|\leq m} c_\beta \partial^\beta S_n\left(|x - x_0|\right) + q(x), \end{aligned} \tag{4.1}$$

where $*$ denotes the convolution product and q is a biharmonic function in Ω. A comparison between Equation (4.1) and the statement in Theorem 3.1 leads to the conclusion that α is the constant c_0 and

$$g(x) = \sum_{1 \le |\beta| \le m} c_\beta \partial^\beta S_n \left(|x - x_0|\right). \tag{4.2}$$

Since this was obtained under the assumption that u can be extended to a distribution in Ω, and since any derivative of S_n defines a distribution in $\mathbb{R}^n$, we have therefore proved.

***Theorem* 4.1** — *Let Ω be an open set in $\mathbb{R}^n$ and $x_0 \in \Omega$. If u is a biharmonic function in $\Omega \backslash \{x_0\}$, then u extends as a distribution in Ω if, and only if, its singular part is a finite linear combination of $S_n \left(|x - x_0|\right)$ and its derivatives.*

Remark : As a consequence of the above Theorem 4.1 and Theorem 1.1 in Futamma and Mizuta [5] we can state: Let u be biharmonic on $\Omega \setminus \{x_0\}$ in $\mathbb{R}^n$. For some real number s, suppose $\liminf_{r \to 0} r^s \int_{|x-x_0|=r} |u(x)| d\sigma(x) = 0$ where $d\sigma$ refers to the surface integral. Then u extends as a biharmonic distribution on Ω; if $n \ge 4 + s$, then u actually extends as a biharmonic function on Ω.

Examples

1. E_n satisfies

$$\Delta^2 E_n = \Delta \delta = \sum_{k=1}^{n} \partial_k^2 \delta$$

and is therefore biharmonic in $\mathbb{R}^n \backslash \{0\}$. According to the representation given in Theorem 3.1, both α and q vanish and $g(x) = E_n(x) = \sum_{k=1}^n \partial_k^2 S_n(x)$.

2. The function $x_k E_n(|x|)$ is biharmonic in $\mathbb{R}^n \backslash \{0\}$ for all $k \in \{1, ..., n\}$, since

$$\begin{aligned} \Delta(x_k E_n) &= 2\partial_k E_n + x_k \delta = 2\partial_k E_n \\ \Delta^2(x_k E_n) &= 2\partial_k \delta. \end{aligned}$$

Here again both α and q vanish and $g(x) = x_k E_n(x) = 2\partial_k S_n(x)$.

3. The function $|x|^2 E_n$, on the other hand, is also biharmonic in $\mathbb{R}^n \backslash \{0\}$, with a singularity of the same order as S_n at $x = 0$ when $n \neq 4$. Here we have

$$\begin{aligned} \Delta^2(|x|^2 E_n) &= \Delta \left(2nE_n + 4\sum_{k=1}^{n} x_k \partial_k E_n + |x|^2 \delta \right) \\ &= 2n\delta + 8\sum_{k=1}^{n} \partial_k^2 E_n + 4\sum_{k=1}^{n} x_k \partial_k \delta \\ &= 2(4-n)\delta. \end{aligned}$$

Consequently $\alpha = 2(4-n)$, whereas both g and q vanish.

But we know, of course, that the singularity of a biharmonic function in Ω need not be of finite order, as in the case of a harmonic function with an essential singularity at x_0. In that case, we would expect the finite sum over $|\beta| \leq m$ in the right-hand side of (4.1) to be replaced by an infinite sum which, in fact, represents the principal part of the Laurent series of u about x_0. Such a function cannot be extended as a distribution to Ω.

5. Biharmonic Function Generated by a Distribution with Compact Support

In this section we consider the case where the biharmonic function u, instead of having a point singularity $\{x_0\}$, has its singularity in a compact set in Ω.

If u is a biharmonic function in $\Omega \backslash K$ which extends to a distribution in Ω, then $\Delta^2 u$ is a distribution in Ω with compact support K and is therefore of finite order, say m. Hence (see [9]), given any compact neighbourhood ω of K in Ω, there is a family of (signed) Radon measures $\{\mu_\beta : |\beta| \leq m\}$ in $\mathbb{R}^n$ such that

$$\Delta^2 u = \sum_{|\beta| \leq m} \partial^\beta \mu_\beta \tag{5.1}$$

with supp $\mu_\beta \subset \omega$ for all $|\beta| \leq m$. The solution of (5.1) is given by

$$u = \sum_{|\beta| \leq m} \partial^\beta S_n * \mu_\beta + q \tag{5.2}$$

for some biharmonic function q in Ω.

In the expression (5.2) the sum $\sum_{|\beta|\leq m} \partial^\beta S_n * \mu_\beta$ is of course the singular component of u. The first term in this sum,

$$S_n * \mu_0(x) = \int_{\mathbb{R}^n} S_n(|x-\xi|)d\mu_0(\xi),$$

is biharmonic in $\mathbb{R}^n \backslash \{\text{supp}\mu_0\}$. Since this integral is over a compact set, we can differentiate inside the integral to obtain

$$\begin{aligned}\Delta(S_n * \mu_0)(x) &= \int_{\mathbb{R}^n} E_n(|x-\xi|)d\mu_0(\xi) \\ &= \mu_0\left(\mathbb{R}^n\right) E_n(|x|) + o\left(|x|^{2-n}\right) \quad \text{as } |x| \to \infty.\end{aligned}$$

Hence $S_n * \mu_0$ is the term which carries the flux of Δu at ∞.

Remark : The case when the distribution $\Delta^2 u$ is of order 0 is therefore of special significance, as it corresponds to the situation characterized by Bôcher's theorem for $K = \{x_0\}$. It is also the case which is called for in most physical applications, a typical example of which is given below.

6. Application to a Boundary-Value Problem

When a thin, elastic plate in the xy plane, supported at its edges, is subjected to a normal pressure load p, the resulting deflection u, according to the linear theory of elastic plates (see [8], for example), satisfies the differential equation

$$D\left(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) = p(x,y),$$

where D is the flexural rigidity constant for the plate. Setting $p/D = \mu$, and assuming that the plate, which occupies the region Ω, is simply supported at the boundary, we arrive at the boundary-value problem

$$\begin{aligned}\Delta^2 u &= \mu \quad in\, \Omega, \\ u &= \Delta u = 0 \quad on\, \partial\Omega.\end{aligned} \tag{6.1}$$

This problem can be solved explicitly for certain simple domains, such as a rectangle, by separation of variables and the solution is expressed as a Fourier series. But this method does not work so well for other shapes, even for a simple circular plate. It also relies on the feasibility of expressing μ as a Fourier series.

A more effective approach to solving the boundary-value problem (6.1) is to construct the Green's function for Ω, and then use the Green's formula to express u as the integral over Ω of the convolution product of the Green's function with μ. This implicitly requires setting up an appropriate Hilbert space structure to handle the Green's formula. But we can follow the procedure outlined above for representing biharmonic functions, and thereby avoid these limitations.

We take Ω to be a bounded domain in $\mathbb{R}^n$ which is regular for the Dirichlet problem, in the sense that, given a continuous function f on $\partial\Omega$, there is a unique harmonic function h in Ω such that $h(x) \to f(\xi)$ as $x \to \xi$ for every $\xi \in \partial\Omega$. We shall also assume that μ is a Radon measure in $\mathbb{R}^n$ with compact support K in Ω.

The (distributional) solution of $\Delta^2 u = \mu$ in $\mathbb{R}^n$ is given by

$$u = S_n * \mu + q, \tag{6.2}$$

where q is a biharmonic function in $\mathbb{R}^n$. With S_n biharmonic in $\mathbb{R}^n\backslash\{0\}$ and locally integrable in $\mathbb{R}^n$, the function

$$S_n * \mu(x) = \int_{\mathbb{R}^n} S_n(|\xi - x|) d\mu(\xi)$$

is well defined in $\mathbb{R}^n$ and biharmonic (and continuous) in $\mathbb{R}^n\backslash K$. As $|x| \to \infty$ it behaves like $\alpha S_n(|x|)$, where $\alpha = \mu(K)$.

Applying the Laplacian operator to equation (6.2) gives

$$\Delta u = E_n * \mu + h_1, \tag{6.3}$$

where $h_1 = \Delta q$ is a harmonic function in $\mathbb{R}^n$. Here also $E_n * \mu$ is harmonic, and hence continuous, in $\mathbb{R}^n\backslash K$ and behaves like $\alpha E_n(|x|)$ as $|x| \to \infty$. Let $h_1' = h_1\chi_\omega$ where ω is a relatively compact domain $\supset \overline{\Omega}$. Then the convolution

$E_n * h_1'$ is well defined and satisfies $\Delta(E_n * h_1') = h_1' = \Delta q$ in ω. There is, therefore, a harmonic function h_2 in ω such that

$$q = E_n * h_1' + h_2 \quad \text{in } \omega. \tag{6.4}$$

Now the continuity of u and $S_n * \mu$ on $\bar{\Omega} \backslash K$ implies, in view of Equation (6.2), the continuity of q, and hence of h_2 on $\bar{\Omega}$.

In the special case when Ω is the ball $B = \{x \in \mathbb{R}^n : |x| < R\}$, the harmonic extensions of $-E_n * \mu|_{\partial\Omega}$ and $-(S_n * \mu + E_n * h_1)|_{\partial\Omega}$ from the sphere ∂B to $\bar{B}$ are readily obtained from the Poisson's integral formula [3].

***Theorem* 6.1** — *Let Ω be a bounded domain in $\mathbb{R}^n$, regular for the Dirichlet problem. Let μ be a Radon measure in $\mathbb{R}^n$ with compact support $K \subset \Omega$. Let f and g be two real-valued continuous functions on $\partial\Omega$. Then, there exists a unique u on $\overline{\Omega}$ such that $\Delta^2 u = \mu$ on Ω, $\Delta u = f$ on $\partial\Omega$ and $u = g$ on $\partial\Omega$.*

PROOF : The (distributional) solution of $\Delta^2 v = \mu$ in $\mathbb{R}^n$ is given by $v = S_n * \mu + q$, where q is a biharmonic function in $\mathbb{R}^n$. Since S_n is biharmonic on $\mathbb{R}^n \setminus \{0\}$ and is locally integrable, $S_n * \mu(x) = \int\limits_{\mathbb{R}^n} S_n(\xi - x) d\mu(\xi)$ is well-defined on $\mathbb{R}^n$ and biharmonic (hence continuous) on $\mathbb{R}^n \setminus K$. As $|x| \to \infty$, it behaves like $\alpha S_n(x)$ where $\alpha = \mu(K)$.

Let h_2 be harmonic on Ω such that $h_2 = f - \Delta v$ on $\partial\Omega$. Set $v_1 = \Delta v + h_2$ in Ω. Then v_1 is harmonic in $\Omega \setminus K$ and $v_1 = f$ on $\partial\Omega$. Let v_2 be a continuous function with compact support in $\mathbb{R}^n$ and $v_2 = h_2$ on $\overline{\Omega}$. Then $b = E_n * v_2$ is well-defined on $\mathbb{R}^n$ and $\Delta b = v_2$ in $\mathbb{R}^n$ and hence $\Delta b = h_2$ in Ω is harmonic in Ω. That is, b is biharmonic on Ω.

Let $u_1 = v + b$. Then $\Delta^2 u_1 = \Delta^2 v = \mu$ on Ω, and in $\mathbb{R}^n$

$$\begin{aligned} \Delta u_1 &= \Delta v + v_2 \\ &= \Delta v + h_2 \text{ on } \overline{\Omega} \\ &= f \text{ on } \partial\Omega. \end{aligned}$$

Note that u_1 is continuous on $\mathbb{R}^n \setminus K$. Let h_1 be harmonic on Ω such that $h_1 = g - u_1$ on $\partial\Omega$. Write $u = u_1 + h_1$. Then u is continuous on $\overline{\Omega} \setminus K$, $u = g$ on $\partial\Omega$, $\Delta^2 u = \Delta^2 u_1 = \mu$ on Ω, and $\Delta u = \Delta u_1 = f$ on $\partial\Omega$.

To prove the uniqueness of u, assume that q is a real-valued function on $\overline{\Omega}$ such that $\Delta^2 q = \mu$ on Ω, $\Delta q = f$ on $\partial\Omega$ and $q = g$ on $\partial\Omega$. Write $p = u - q$. Then $\Delta^2 p = 0$ on Ω, so that p is biharmonic on Ω, hence Δp is harmonic on Ω such that $\Delta p = 0$ on $\partial\Omega$, so that $\Delta p = 0$ on $\overline{\Omega}$. That is, p is harmonic on Ω and $p = 0$ on $\partial\Omega$, hence $p = 0$ on $\overline{\Omega}$. Thus $q = u$ on $\overline{\Omega}$. ■

7. SOME GENERALIZATIONS

The representation of a biharmonic function has been shown to depend in an essential way on the well known properties of harmonic functions. The same argument above may be used to represent functions of higher order of harmonicity, that is, solutions of $\Delta^m u = 0$ for integer values of $m > 2$. These are sometimes referred to as *polyharmonic functions* of order m, though this term is used in a slightly different sense in [4].

If $\Delta^m u = 0$ in $\Omega \backslash K$ then $\Delta^{m-1} u$ is harmonic and $\Delta^{m-2} u$ is biharmonic in $\Omega \backslash K$, and so Theorems 2.1 and 2.2 may be used to represent these functions. When K is a single point x_0, this leads in a natural way to the decomposition

$$u(x) = \alpha E_{n,m}(|x - x_0|) + h(x) + r(x) \quad \text{in } \Omega \backslash \{x_0\}, \tag{7.1}$$

where $E_{n,m}$ is the fundamental singularity of Δ^m, that is, it satisfies $\Delta^m E_{n,m} = \delta$, whereas h is polyharmonic in $\Omega \backslash \{x_0\}$ and r is polyharmonic in Ω, both of order m. Following the procedure of Section 4, the function h may be expressed as a linear combination of derivatives of $E_{n,m}$, and the representation (7.1) is unique up to an additive polyharmonic function of order $m - 1$ in $\mathbb{R}^n$.

Similarly if u is a polyharmonic function of order m in $\Omega \backslash K$ which extends to a distribution in Ω, then the formula (5.2) applies with the obvious modifications, namely that S_n be replaced by $E_{n,m}$ and q be polyharmonic of order m in Ω.

8. Biharmonic Vertex Singularity in the Discrete Case

In this section, we consider the discrete analogue of the above results in the context of an infinite graph, consisting of a countably infinite number of vertices and a countably infinite number of edges.

Let X be an infinite tree, connected and locally finite and without any terminal vertices [2]. To any pair of vertices x and y is associated a number $t(x,y) \geq 0$ such that $t(x,y) > 0$ if and only if $[x,y]$ is an edge in X (that is x and y are neighbours). We do not suppose $t(x,y) = t(y,x)$.

If E is a subset of X, we say that $x \in E$ is an interior vertex of E if all the neighbours of x are also in E. We denote the set of interior vertices of E by $\overset{\circ}{E}$. For a real-valued function u on E, u is said to be harmonic if $\Delta u(x) = \sum_y t(x,y)[u(y) - u(x)] = 0$ for every $x \in \overset{\circ}{E}$. If v is a real-valued function on X for which $\Delta^2 v(x) = \Delta(\Delta v)(x) = 0$ at every $x \in \overset{\circ}{E}$, then v is said to be biharmonic on E.

For any $a \in X$, we can construct a function $q_a(x)$ on X such that $(-\Delta) q_a(x) = \delta_a(x)$ for any $x \in X$ [2, Theorems 3.2.6 and 3.4.5]; $q_a(x)$ can be uniquely fixed if we impose the restriction that q_a is a potential when X is hyperbolic and q_a is a pseudo-potential if X is parabolic. Consequently, if f is any real-valued function on X such that $f = 0$ outside a finite set, then $g(x) = \sum_y f(y) q_y(x)$ is a well-defined function on X for which $(-\Delta) g(x) = f(x)$ for every $x \in X$. Moreover since X has no terminal vertices, there exists $Q_a(x)$ on X such that $(-\Delta) Q_a(x) = q_a(x)$.

***Theorem* 8.2** — *Let E be a finite subset of X and $a \in \overset{\circ}{E}$. Let $u(x)$ be a real-valued function on X such that $\Delta^2 u(x) = 0$ for every $x \in \overset{\circ}{E} \setminus \{a\}$. Then there exist a function $B(x)$ such that $\Delta^2 B(x) = 0$ for every $x \in \overset{\circ}{E}$ and two uniquely determined constants λ and μ such that $u(x) = B(x) + \lambda Q_a(x) + \mu q_a(x)$ for every $x \in E$.*

Proof : Let $\Delta u(x) = h(x)$ for $x \in \overset{\circ}{E} \setminus \{a\}$. Then $\Delta h(x) = 0$ for every

$x \in \overset{\circ}{E} \setminus \{a\}$. Let $\Delta h(a) = -\lambda$. Write $v(x) = h(x) + \lambda q_a(x)$ for $x \in \overset{\circ}{E}$.

Let $b(x)$ be defined on E such that $\Delta b(x) = v(x)$ for $x \in E$ so that $\Delta^2 b(x) = 0$ for $x \in \overset{\circ}{E}$, that is, $b(x)$ is biharmonic on E. Then for $x \in \overset{\circ}{E} \setminus \{a\}$, $\Delta b(x) = \Delta u(x) - \lambda \Delta Q_a(x)$. Consequently, $f(x) = u(x) - [b(x) + \lambda Q_a(x)]$, which is well-defined on E, satisfies the condition $\Delta f(x) = 0$ for $x \in \overset{\circ}{E} \setminus \{a\}$. Hence $f(x) = \mu q_a(x) + g(x)$ for $x \in E$, where μ is a constant and $\Delta g(x) = 0$ for $x \in \overset{\circ}{E}$.

Thus, on E,

$$\begin{aligned} u(x) &= b(x) + \lambda Q_a(x) + \mu q_a(x) + g(x) \\ &= B(x) + \lambda Q_a(x) + \mu q_a(x), \end{aligned}$$

where $\Delta^2 B(x) = 0$ for every $x \in \overset{\circ}{E}$.

Uniqueness: Suppose $u(x) = B'(x) + \lambda' Q_a(x) + \mu' q_a(x)$ is another such decomposition. Then $-(\lambda - \lambda')q_a(x) + \Delta(B - B')(x) = 0$ for every $x \in \overset{\circ}{E}$. Since $\Delta(B - B')$ is harmonic on E, we conclude that $\lambda = \lambda'$. Now $B(x) - B'(x) = -(\mu - \mu')q_a(x)$ so that $\Delta(B - B') = (\mu - \mu')\delta_a$ is harmonic on E. Hence, we conclude $\mu = \mu'$. ■

Remark : An infinite network X is called a *bipotential network* if there exist two positive potentials p and q such that $\Delta q = -p$ [2, p. 124]. In a bipotential network X, if $G_y(x)$ is the Green potential with harmonic support $\{y\}$, then there exists a unique potential $Q_y(x)$, called the *biharmonic Green potential*, such that $\Delta Q_y(x) = -G_y(x)$ [2, Corollary 5.2.5]. Consequently, in the above Theorem 8.1, if X is a bipotential tree, then not only the constants λ and μ but also the biharmonic function $B(x)$ is uniquely determined.

ACKNOWLEDGEMENT

This research was sponsored by the Distinguished Scientist Fellowship Program at King Saud University, Riyadh, Saudi Arabia.

References

1. M. A. Al-Gwaiz, *Theory of Distributions,* Marcel Dekker, 1992.
2. V. Anandam, *Harmonic Functions and Potentials on Finite or Infinite Networks,* UMI Lecture Notes 12, Springer, 2011.
3. V. Anandam and M. A. Al-Gwaiz, Global representation of harmonic and biharmonic functions, *Potential Analysis*, **6** (1997), 207-214.
4. N. Aronszajn, T. M. Creese and L. J. Lipkin, *Polyharmonic Functions,* Clarendon Press, Oxford, 1983.
5. T. Futamma and Y. Mizuta, Isolated singularties of super-polyharmonic functions, *Hokkaido Math. Journal*, **33** (2004), 675-695.
6. L. L. Helms, *Introduction to Potential Theory*, 2nd edition, Springer-Verlag, 2009.
7. M. Hervé, *Transformation de Fourier et distributions*, Presses Universitaries de France, 1986.
8. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York, 1944.
9. F. Treves, *Topological Vector Spaces*, *Distributions and Kernels*, Academic Press, New York, 1967.