Relations

Mongi BLEL

King Saud University

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Relations

The topic of this chapter is relations, it is about having 2 sets, and connecting related elements from one set to another. There is three important type of relations:
functions, equivalence relations and order relations. In this chapter, equivalence and order relations are only considered.

Definition

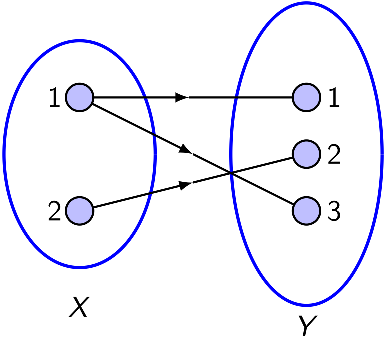
Let X and Y be two sets. A binary relation R from X to Y is a subset of the Cartesian product $X \times Y$. Given $x, y \in X \times Y$, we say that x is related to y by R, also written (xRy) if and only if $(x, y) \in R$.

Definition

Let R be a binary relation from X to Y.
the set $D(R) = \{x \in X;\ (x, y) \in R\}$ is called the domain of the relation. The set $R(R) = \{y \in Y;\ (x, y) \in R\}$ is called the range of the relation.

Example

Let $X = \{1, 2\}$ and $Y = \{1, 2, 3\}$, and the relation is given by $(x, y) \in R \iff x - y$ is even.
$X \times Y = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$ and
$R = \{(1, 1), (1, 3), (2, 2)\}$.
To illustrate this relation we use the following diagram:



Definition

A relation on a set X is a relation from X to X. In other words, a relation on a set X is a subset of $X \times X$. (Relation of the same set is called also homogeneous relation)

Example

Let $X = \{1, 2, 3, 4\}$ and $R = \{(a, b);\ a \text{ divides } b\}$. Then

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Definition

Let R be a relation from the set X to the set Y.The inverse relation R^{-1} from Y to X is defined by:

$$R^{-1} = \{(y,x) \in Y \times X,\ (x,y) \in R\}.$$

(The inverse relation R^{-1} is also called the transpose or the converse relation of R and denoted also R^T).

1. Consider the sets $X = \{2, 3, 4\}$, $Y = \{2, 6, 8\}$, with the relation $(x, y) \in R$ if and only if x divides y.

$$X \times Y = \{(2,2),(2,6),(2,8),(3,2),(3,6),(3,8),(4,2),(4,6),(4,8)\},$$

$$R = \{(2,2),(2,6),(2,8),(3,6),(4,8)\},$$

$$R^{-1} = \{(2,2),(6,2),(8,2),(6,3),(8,4)\}.$$

$(y, x) \in R^{-1}$ if and only if y is a multiple of x.

2. The **identity relation** defined on a set X is defined by $I = \{(x, x);\ x \in X\}$.
3. The **universal relation** R from X to Y is defined by $R = X \times Y$.
4. Let $X = \mathbb{Z}$ and R the relation defined by: $mRn \iff m^2 - n^2 = m - n$. Since $m^2 - n^2 = (m-n)(m+n)$, then $mRn \iff m = n$ or $m + n = 1$. Then

$$R = \{(m, m), (m, 1-m);\ m \in \mathbb{Z}\}.$$

Boolean matrix of relation

If $X = \{x_1, \ldots, x_n\}$ and $X = \{y_1, \ldots, y_m\}$ are finite sets and R a binary relation from X to Y, we represent the relation R by the following matrix: (called the Boolean matrix of R)

$$M_R = \begin{pmatrix} x_1 R x_y & x_1 R y_2 & \ldots & x_1 R y_m \\ x_2 R y_1 & x_2 R y_2 & \ldots & x_2 R y_m \\ \vdots & \vdots & & \vdots \\ x_n R y_1 & \ldots & \ldots & x_n R y_m \end{pmatrix},$$

where $x_j R y_k = 1$ if $(x_j, y_k) \in R$ and 0 otherwise.

For example if $X = \{2, 3, 4\}$, $Y = \{2, 6, 8\}$, and the relation R defined by: $(x, y) \in R$ if and only if x divides y.
The relation R is represented by the following matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix which represents R^{-1} is the transpose of this matrix.

Definition

Let R, S be two relations in $X \times Y$. The relations $R \cup S$ and $R \cap S$ are called respectively the union and the intersection of these relations.

Example

Let R_1 and R_2 the relations on the set $X = \{a, b, c\}$ represented respectively by the matrices

$$M_{R_1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_{R_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

$R_1 = \{(a, a), (a, c), (b, a), (c, b)\}$,
$R_2 = \{(a, a), (a, c), (b, b), (b, c), (c, a)\}$.
$R_1 \cap R_2 = \{(a, a), (a, c)\}$,
$R_1 \cup R_2 = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b)\}$.
$R_1 - R_2 = \{(b, a), (c, b)\}$, $R_2 - R_1 = \{(b, b), (b, c), (c, a)\}$.

The matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$ are respectively:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Composition of Relations

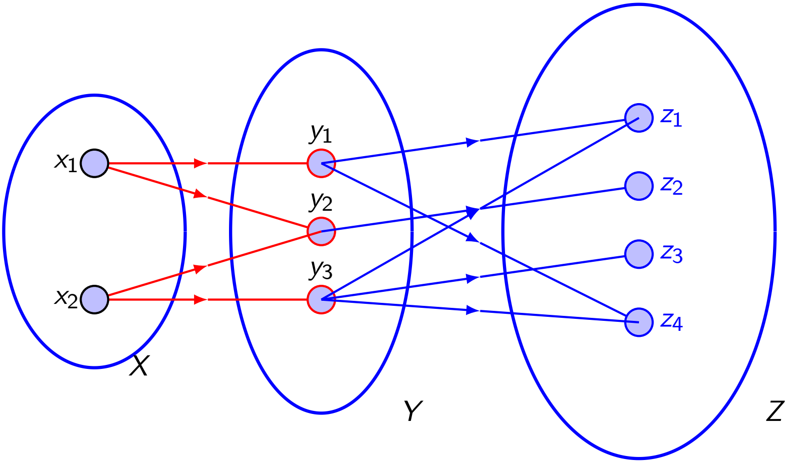
Definition

Given two relations $R \in X \times Y$ and $S \in Y \times Z$, the composition of R and S is the relation on $X \times Z$ defined by:

$$S \circ R = \{(x, z) \in X \times Z,\ \exists y \in Y, xRy, ySz\}.$$

Example

$$
\begin{aligned}
&X=\{x_1,x_2\},\ Y=\{y_1,y_2,y_3\},\ Z=\{z_1,z_2,z_3,z_4\},\\
&R=\{(x_1,y_1),(x_1,y_2),(x_2,y_2),(x_2,y_3)\},\\
&S=\{(y_1,z_1),(y_1,z_4),(y_2,z_2),(y_3,z_1),(y_3,z_3),(y_3,z_4)\},\\
&S\circ R=\{(x_1,z_1),(x_1,z_2),(x_1,z_4),(x_2,z_1),(x_2,z_2),(x_2,z_3),(x_2,z_4)\}.
\end{aligned}
$$


x_1
x_2
X
y_1
y_2
y_3
Y
z_1
z_2
z_3
z_4
Z

The matrices of the relations R and S are respectively

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad M_S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

The matrix representing $S \circ R$ is:

$$M_{S \circ R} = M_R . M_S = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The product of matrices is the Boolean product defined as the following: if $A = (a_{j,k})$ is a Boolean matrix of degree (m, n) and $B = (b_{j,k})$ is a Boolean matrix of degree (n, p), $A.B = (c_{j,k})$ is the Boolean matrix of degree (m, p) defined by:

$$C_{j,k} = \max\{a_{j,1} b_{1,k}, a_{j,2} b_{2,k}, \ldots, a_{j,n} b_{n,k}\}.$$

Example

Let R be the relation from the set of names to the set of telephone numbers and let S be the relation from the set of telephone numbers to the set of telephone bills. The relations R and S are defined by the below tables.
Then the relation $S \circ R$ is a relation from the set of names to the set of telephone bills.

Table of the relation R	
Ali	104105106, 105325118, 104175100
Ahmed	105315307, 104137116, 107325112
Salah	107107121
Salem	104271216, 105145146

Table of the relation S	
104105106	735
105325118	245
104175100	535
105315307	250
104137116	1250
107325112	275
107107121	2455
104271216	445
105145146	1215

Table of the relation $S \circ R$	
Ali	1515
Ahmed	1775
Salah	2455
Salem	1660

Theorem

Let R_1 be a relation from X to Y and R_2 a relation from Y to Z. Then $(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$.

Proof:

$$R_2 \circ R_1 = \{(x, z) \in X \times Z;\ \exists y \in Y,\ (x, y) \in R_1,\ (y, z) \in R_2\}$$

$$\begin{aligned}(R_2 \circ R_1)^{-1} &= \{(z, x) \in Z \times X;\ \exists y \in Y,\ (x, y) \in R_1,\ (y, z) \in R_2\} \\ &= \{(z, x) \in Z \times X;\ \exists y \in Y,\ (y, x) \in R_1^{-1},\ (z, y) \in R_2^{-1}\} \\ &= R_1^{-1} \circ R_2^{-1}\end{aligned}$$

Definition

Let R be a relation on the set X. The powers R^n, $n \in \mathbb{N}$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Example

If $X = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (1, 3), (2, 1), (3, 4)\}$. Then
$R^2 = \{(1, 1), (1, 4), (2, 2), (2, 3)\}$,
$R^3 = \{(1, 2), (1, 3), (2, 1), (2, 4)\}$.

$$M_R = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_{R^2} = M_R^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Representing Relations Using Digraphs

We have shown that a relation can be represented by listing all of its ordered pairs or by using a Boolean matrix. There is another representation. Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow. We use such pictorial representations when we think of relations on a finite set as directed graphs, or digraphs.

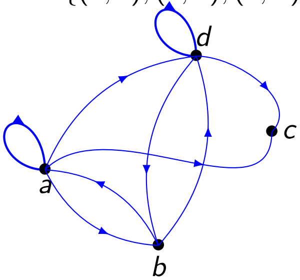
Definition

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

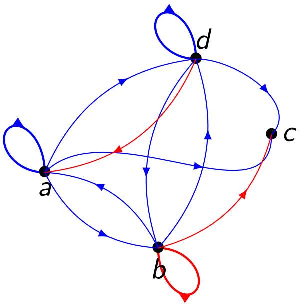
When a relation R is defined on a set X, the arrow diagram of the relation can be modified so that it becomes a directed graph.
Instead of representing X as two separate sets of points, represent X only once, and draw an arrow from each point of X to each $R-$related point.
If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

Example

Let $X = \{a, b, c, d\}$ and
$R = \{(a, a), (a, b), (a, d), (b, a), (b, d), (d, d), (d, b), (d, c)\}$

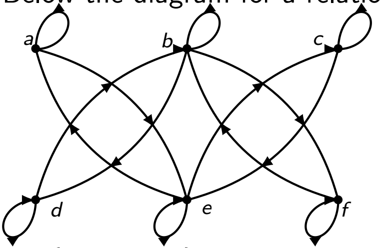


The digraph of the relation R^2



Example

Below the diagram for a relation R on a set X.



$X = \{a, b, c, d, e, f\}$,
$R = \{(a, a), (a, e), (b, b), (b, d), (b, f), (c, c), (c, e), (d, b), (d, d), (e, a), (e, c), (e, e), (f, b), (f, f)\}$

Definition

A relation R on a set X is called reflexive if every element of X is related to itself: $\forall x \in X$, xRx.
If X is finite, R is reflexive if and only if $I \subset R$.

Example

If $X = \mathbb{Z}$ and the relation R is defined by $xRy \iff x - y \equiv 0[3]$.
This relation is reflexive.

Definition

A relation R on a set X is called symmetric if $(x, y) \in R$ then $(y, x) \in R$: $\forall x, y \in X$, $xRy \Rightarrow yRx$.
R is symmetric if and only if $R^{-1} = R^T = R$.

Example

If $X = \mathbb{Z}$ and the relation R is defined by: $xRy \iff x - y \equiv 0[3]$.
This relation is symmetric.

Definition

A relation R on a set X is called transitive if: $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$: $\forall x, y, z \in X$, $xRy \wedge yRz \Rightarrow xRz$.

Remarks

1. A relation R on a set X is reflexive if and only if the diagonal relation on X is a subset in R. (The diagonal relation on X is the relation $I = \{(x, x);\ x \in X\}$).
2. A relation R on a set X is symmetric if and only if $R^{-1} = R$.
3. A relation R on a set A is transitive if and only if $R^n \subset R$ for all $n \geq 2$.
4. A relation R on a set A is transitive if and only if $R^2 \subset R$.

Definition

A relation R on a set X is called an equivalence relation if R is reflexive, symmetric and transitive.
The equivalence class of a in X is

$$[a] = \{x \in X, aRx\}.$$

R is an equivalence relation if and only if $R^2 \subset R \subset R^T$ and $I \subset R$.

Example

The relation $\equiv [\mathrm{mod}\ n]$ is an equivalence relation on $\mathbb{Z}$.

• It is reflexive: $x \equiv x[\mathrm{mod}\ n]$ is always true.

• It is symmetric: $x \equiv y[\mathrm{mod}\ n]$ means that $x = qn + y$ for some integer q, thus $y = -qn + x$ and $y \equiv x[\mathrm{mod}\ n]$.

• It is transitive: if $x \equiv y[\mathrm{mod}\ n]$ and $y \equiv z[\mathrm{mod}\ n]$ then we have $x = qn + y$ and $y = rn + z$ thus $x = qn + y = n(q + r) + z$ and $x \equiv z[\mathrm{mod}\ n]$.

The equivalence class of 0 is the multiples of n:

$$[0] = \{kn,\ k \in \mathbb{Z}\}.$$

Theorem

Let R be an equivalence relation on a set X. These statements for elements a and b of X are equivalent:

1. aRb
2. $[a] = [b]$
3. $[a] \cap [b] \neq \emptyset$.

Definition

A collection of subsets X_j, $j \in I$ (where I is an index set) forms a partition of X if $X_j \neq \emptyset$ for $j \in I$, $X_j \cap X_k = \emptyset$ and $\cup_{j \in I} X_j = X$.

Theorem

The equivalence classes of an equivalence relation R on a set X form a partition of X. Conversely, given a partition $\{A_j;\ j \in I\}$ of the set X, there is an equivalence relation R that has the sets A_j , $j \in I$ as its equivalence classes.

Definition

Given $(X_j)_{j\in I}$ a partition of X. The equivalence relation R on X related to this partition is the relation defined by:

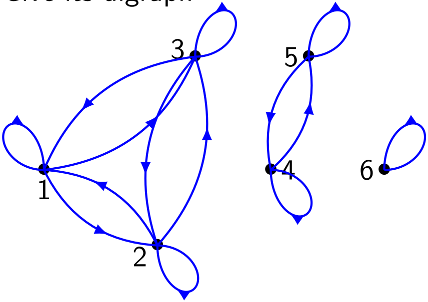
$$xRy \iff \exists j \in I;\ x, y \in X_j.$$

Example

Let $X = \{0, 1, 2, 3, 4, 5\}$ and the partition of X: $\{0, 3, 4\}$, $\{1, 5\}$, $\{2\}$. The equivalence relation R induced by this partition is

$$\begin{aligned} R &= \{(0,0),(0,3),(3,0),(0,4),(4,0),(3,3),(3,4),(4,3),\\ &\qquad (4,4),(1,1),(1,5),(5,1),(5,5),(2,2)\}. \end{aligned}$$

Let R be the relation produced by the partition $X_1 = \{1, 2, 3\}$, $X_2 = \{4, 5\}$ and $X_3 = \{6\}$ of $X = \{1, 2, 3, 4, 5, 6\}$.
Give its digraph



Definition

A relation R on a set X is antisymmetric if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$: $\forall x, y \in X$, $xRy \wedge yRx \Rightarrow x = y$.
R is antisymmetric if and only if $R \cap R^{-1} \subset I$.

Example

If $X = \mathbb{Z}$ and the relation R is defined by: $xRy \iff x \leq y$. This relation is antisymmetric

Definition

A relation R on a set X is a partial order if R is reflexive, antisymmetric and transitive.
A set X together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (X, R).

Example

If $X = \mathbb{Z}$ and the relation R is defined by: $xRy \iff x \leq y$. This relation is a partial order.

Example

Suppose that a relation R on a set is represented by the following matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The relation R is reflexive because all the diagonal elements of this matrix are equal to 1.
The relation R is symmetric because $R^T = R$.
The relation R is not transitive and not antisymmetric.

Theorem

Let R be a partial order relation on X, then R^{-1} is also a partial order relation on X.

Definition

Let $(X, \leq)$ be a partial ordering set. The elements $a, b \in X$ are called comparable if either $a \leq b$ or $b \leq a$.
If neither $a \leq b$ nor $b \leq a$, we say that a and b are incomparable.
If any two elements in X are comparable, we say that the ordered set $(X, \leq)$ is total or linearly ordered set and the relation $\leq$ is called a total order or a linear order. A totally ordered set is also called a chain.

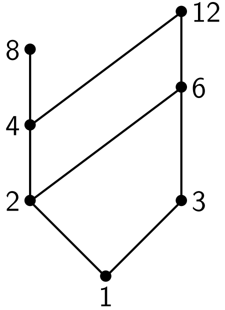
Example

Let $(\mathbb{N}, R)$ be the ordered set defined by the relation nRm if n divides m. The integers 3 and 9 comparable but 2 and 3 are not.

Let (X, R) be a finite poset. Many edges in the directed graph for a finite poset do not have to be shown because they must be present. The relation is reflexive, we do not have to show these loops because they must be present.
The relation is transitive, we do not have to show those edges that must be present because of transitivity.
Finally, draw the remaining edges upward and drop all arrows.

Example

The Hasse diagram representing the partial ordering $\{(a, b);\ a \text{ divides } b\}$ on the set $X = \{1, 2, 3, 4, 6, 8, 12\}$.



To obtain a Hasse diagram, proceed as follows:
Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Then eliminate

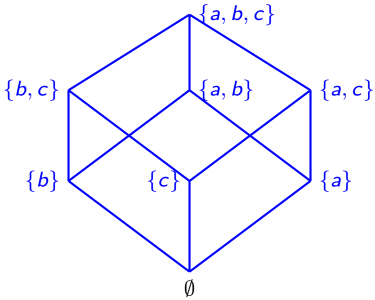
1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

Example

Consider the relation, $\subset$, on the set $\mathcal{P}(a, b, c)$. That is, for all sets U and V in $\mathcal{P}(a, b, c)$,

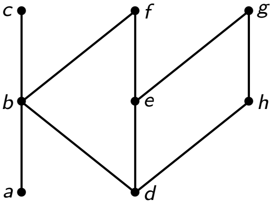
$$U \subset V \iff \forall x \in U,\ x \in V.$$

The Hasse diagram for this relation is:



Example

A partial order R on a set X with the following Hasse diagram.



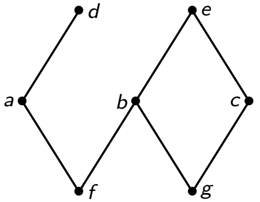
List the elements of R.

$$\begin{aligned} R \quad = \quad & \{(a,a),(a,b),(a,c),(a,f),(d,d),(d,b),(d,e),(d,h),(d,c), \\ & (d,f),(d,g),(b,b),(b,c),(b,f),(e,e),(e,f),(e,g), \\ & (h,h),(h,g),(c,c),(f,f),(g,g)\} \end{aligned}$$

Example

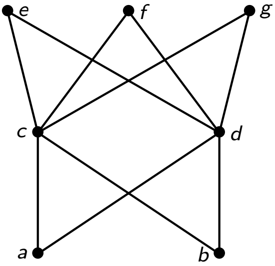
Draw a Hasse diagram of the partial order relation R on $X = \{a, b, c, d, e, f, g\}$ given by

$$\begin{aligned} R \;=\; & \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(g,g),(a,d),(b,e),\\ & (c,e),(f,a),(f,b),(f,d),(f,e),(g,b),(g,c),(g,e)\}. \end{aligned}$$



Example

Example of a poset $(X, \leq)$ which hass the following Hasse diagram.

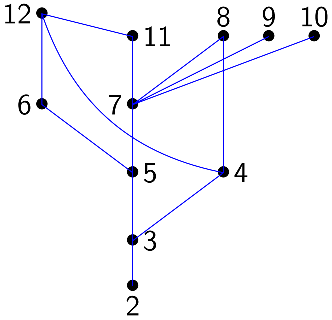


We take the relation R (divides),
$a = 2, b = 3, c = 12, d = 18, e = 180, f = 252, g = 396.$

Example

Let $X = \{n \in \mathbb{N};\ 2 \leq n \leq 12\}$. A partial order relation R on X is defined by mRn if and only if either (m divides n) or (m is prime and $m < n$).

The Hasse diagram



A partial order relation R on $X = \{a, b, c, d, e, f, g\}$ with the following directed graph. Draw its Hasse diagram.