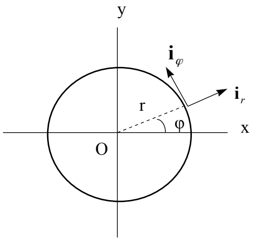
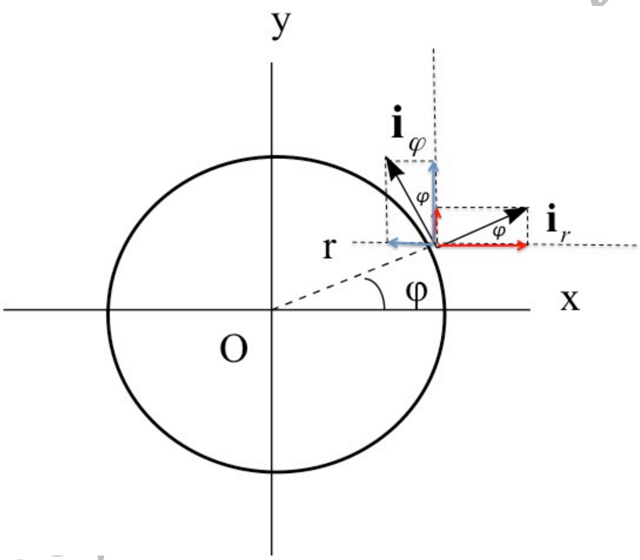
PHYSICS 501
1st HOMEWORK
Dr. V. Lempesis



1. For objects that move in a circle about an origin O, it can be convenient to use the mutually perpendicular unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\varphi$ as shown in figure. Express $\mathbf{i}_r$ and $\mathbf{i}_\varphi$ as a combination of $\mathbf{i}$ and $\mathbf{j}$.

(5 marks)

Solution:



As the figure shows the two vectors $\mathbf{i}_r$ and $\mathbf{i}_\varphi$ can be resolved into two components along the x- and y-directions. From the figure we have:

$$\mathbf{i}_r = i_r \cos\varphi \mathbf{i} + i_r \sin\varphi \mathbf{j} \underset{i_r=1}{\Rightarrow} \mathbf{i}_r = \cos\varphi \mathbf{i} + \sin\varphi \mathbf{j}$$

$$\mathbf{i}_\varphi = -i_\varphi \sin\varphi \mathbf{i} + i_\varphi \cos\varphi \mathbf{j} \underset{i_\varphi=1}{\Rightarrow} \mathbf{i}_\varphi = -\sin\varphi \mathbf{i} + \cos\varphi \mathbf{j}$$

2. Show that $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$.

(5 marks)

Solution:

$$\mathbf{B}\times\mathbf{C}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}=\mathbf{i}\begin{vmatrix}B_y & B_z\\ C_y & C_z\end{vmatrix}-\mathbf{j}\begin{vmatrix}B_x & B_z\\ C_x & C_z\end{vmatrix}+\mathbf{k}\begin{vmatrix}B_x & B_y\\ C_x & C_y\end{vmatrix}=$$

$$\mathbf{i}\left(B_yC_z-B_zC_y\right)-\mathbf{j}\left(B_xC_z-B_zC_x\right)+\mathbf{k}\left(B_xC_y-B_yC_x\right)$$

So

$$\mathbf{A}\cdot\left(\mathbf{B}\times\mathbf{C}\right)=A_x\left(B_yC_z-B_zC_y\right)-A_y\left(B_xC_z-B_zC_x\right)+A_z\left(B_xC_y-B_yC_x\right). \quad (1)$$

We also have

$$\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}=A_x\begin{vmatrix}B_y & B_z\\ C_y & C_z\end{vmatrix}-A_y\begin{vmatrix}B_x & B_z\\ C_x & C_z\end{vmatrix}+A_z\begin{vmatrix}B_x & B_y\\ C_x & C_y\end{vmatrix}= \quad (2)$$

$$A_x\left(B_yC_z-B_zC_y\right)-A_y\left(B_xC_z-B_zC_x\right)+A_z\left(B_xC_y-B_yC_x\right).$$

Comparing (1) and (2) we have that:

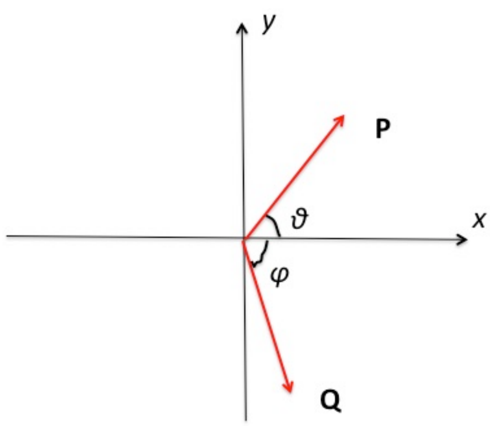
$$\mathbf{A}\cdot\left(\mathbf{B}\times\mathbf{C}\right)=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

3. Using the vectors $\mathbf{P}=\mathbf{i}\cos\theta+\mathbf{j}\sin\theta$, $\mathbf{Q}=\mathbf{i}\cos\phi-\mathbf{j}\sin\phi$, prove the familiar trigonometric identity

$$\cos\left(\theta+\phi\right)=\cos\theta\cos\phi-\sin\theta\sin\phi\,.$$

(5 marks)

Solution:



From the figure we have that:

$$\mathbf{P}\cdot\mathbf{Q} = PQ\cos\left(\vartheta+\varphi\right) \underset{P=Q=1}{=} \cos\left(\vartheta+\varphi\right). \quad (1)$$

Also we have:

$$\mathbf{P}\cdot\mathbf{Q} = P\cos\vartheta Q\cos\varphi - P\sin\vartheta Q\cos\varphi \underset{P=Q=1}{=} \cos\vartheta\cos\varphi - \sin\vartheta\cos\varphi . \quad (2)$$

From (1) and (2) we have:

$$\cos\left(\theta+\phi\right) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

4. Prove that two vectors **A** and **B** must have equal magnitudes if their sum **A+B** is perpendicular (orthogonal) to their difference **A-B**.

(5 marks)

Solution:
If the two vectors are perpendicular then their dot product must be zero.

$$\left(\mathbf{A}+\mathbf{B}\right)\cdot\left(\mathbf{A}-\mathbf{B}\right) = 0 \Rightarrow \mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} = 0 \underset{\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}}{\Rightarrow}$$
$$A^2 - B^2 = 0 \Rightarrow A^2 = B^2 \Rightarrow A = B$$