Math 316

First Midterm Exam 1445, 1st semester

Q1 Prove or disprove each of the following statements:

(a) If a set $\{x_1, x_2, ..., x_n\}$ is orthogonal in an inner product space X, then it is linearly independent.

(b) If $f(x) = \ln x$ and $\rho(x) = \frac{1}{x}$, then $f \in \mathcal{L}^2_\rho(0,1)$.

Q2 Consider the sequence of functions

$$f_n(x) = x^n, \qquad x \in [0,1]$$

(a) Find the limit $f(x)$ of $f_n(x)$ as $n \to \infty$.

(b) Does $f_n(x)$ converge to $f(x)$ uniformly? Justify your answer.

(c) Does $f_n(x)$ converge to $f(x)$ in $\mathcal{L}^2([0,1])$? Justify your answer.

Q3 Consider the eigenvalue problem

$$\begin{aligned} Lu + \lambda u = 0, &\qquad x \in [a,b], \\ u(a) = 0, &\qquad u(b) = 0 \end{aligned} \tag{1}$$

(a) Prove that if L is a self-adjoint operator, then $\lambda \in \mathbb{R}$.

(b) Show that if $L = \left(1+3x^2\right)\frac{d^2}{dx^2} + 6x\frac{d}{dx}$ in problem (1), then L is a self-adjoint operator.

Q4 Consider the eigenvalue problem

$$\begin{aligned} u'' + 2u' + \lambda u = 0, &\qquad x \in [0,1], \\ u(0) = 0, &\qquad u(1) = 0 \end{aligned} \tag{2}$$

(a) Find the eigenvalues and eigenfunctions of problem (2).

(b) Show that L is not a self-adjoint operator.

(c) Transform L into a self-adjoint operator.

(d) Write the orthogonality relation between the eigenfunctions of problem (2).

Good Luck
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