PHYSICS 507 – SPRING 2021
1st HOMEWORK- Solutions
Dr. V. Lempesis

Hand in: Monday 1st of February at 23:59

1. (a) Find the value of the electric flux through the surface of a sphere containing 3 protons and 5 neutrons. $|e| = 1.6\times10^{-19}\,C,\ \varepsilon_0 = 8.85\times10^{-12}\,F/m$.

(2 marks)

(b) Does the size of the sphere matter in the answer of question (a)?

(2 mark)

Solution:

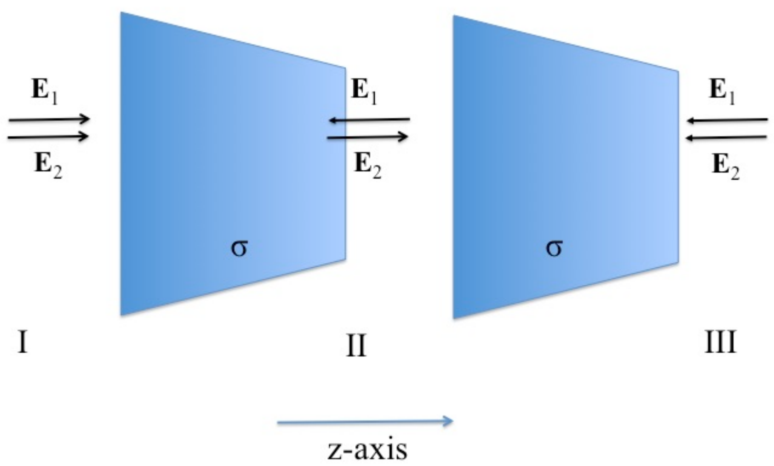
(a) The electric flux through the spherical surface is given by $\Phi = q_{\text{net}}/\varepsilon_0$. The neutrons do not carry electric charge thus they do not contribute to the flux. So

$$\Phi = q_{\text{net}}/\varepsilon_0 = 3|e|/\varepsilon_0 = 5.42\times10^{-8}\ \text{Nm}^2/\text{C}$$

(b) No, the size plays no role.

2. Two infinite parallel planes carry equal and NEGATIVE uniform charge densities σ. Find the field in each of three regions: (i) to the left of both, (ii) between them, (iii) to the right of both. (4 marks)

Solution:



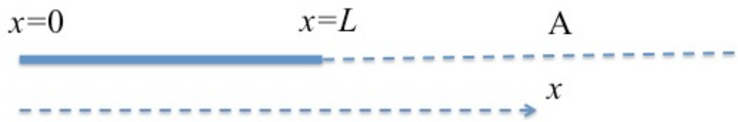
We know that an infinite uniformly charged plane creates an electric field of magnitude $E = |\sigma|/2\varepsilon_0$ (where I use $|\sigma|$ because the charge density is negative). The direction of the fields created by the two planes are shown in the figure. We consider the direction to the right as the positive one. So the total electric field is:

Region (I): $$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{|\sigma|}{2\varepsilon_0}\hat{\mathbf{z}} + \frac{|\sigma|}{2\varepsilon_0}\hat{\mathbf{z}} = \frac{|\sigma|}{\varepsilon_0}\hat{\mathbf{z}}.$$

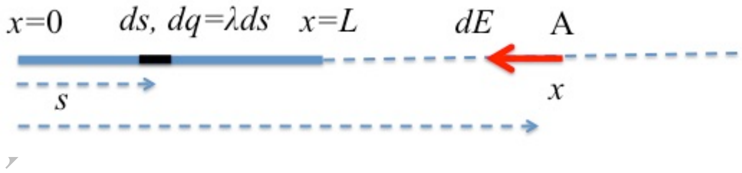
Region (II): $$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = -\frac{|\sigma|}{2\varepsilon_0}\hat{\mathbf{z}} + \frac{|\sigma|}{2\varepsilon_0}\hat{\mathbf{z}} = 0\hat{\mathbf{z}}.$$

Region (III): $$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = -\frac{|\sigma|}{2\varepsilon_0}\hat{\mathbf{z}} - \frac{|\sigma|}{2\varepsilon_0}\hat{\mathbf{z}} = -\frac{|\sigma|}{\varepsilon_0}\hat{\mathbf{z}}$$

3. An infinitely long wire carries **negative** charge with uniform linear charge density λ. Find the electric field at point A at a distance x from the origin. (4 marks).



Solution:
Consider a small part of the wire of length ds and charge $dq = \lambda ds$ which creates at A an elementary electric field dE pointing to the left because the charge distribution is negative. We consider the direction to the right as the positive one.



$$d\mathbf{E} = \frac{dq}{4\pi\varepsilon_0 (x-s)^2}\hat{\mathbf{x}} = -\frac{|\lambda| ds}{4\pi\varepsilon_0 (x-s)^2}\hat{\mathbf{x}}$$

Where I used $|\lambda|$ because the charge is negative. Thus for the total electric field at x we have to integrate **over the charge distribution (i.e on variable *s*, not *x*)**

$$\mathbf{E} = \int_{s=0}^{L} d\mathbf{E} = -\hat{\mathbf{x}} \int_{s=0}^{L} \frac{|\lambda| ds}{4\pi\varepsilon_0 (x-s)^2} = -\hat{\mathbf{x}} \frac{|\lambda|}{4\pi\varepsilon_0} \int_{s=0}^{L} \frac{ds}{(x-s)^2}$$

$$= -\hat{\mathbf{x}} \frac{|\lambda|}{4\pi\varepsilon_0} \left[\frac{1}{x-s} \right]_{s=0}^{L} = -\hat{\mathbf{x}} \frac{|\lambda|}{4\pi\varepsilon_0} \left(\frac{1}{x-L} - \frac{1}{x} \right) = -\hat{\mathbf{x}} \frac{|\lambda|}{4\pi\varepsilon_0} \frac{L}{x(x-L)}$$

The direction is as shown in figure because λ is negative

4. Find the electric charge density inside a sphere where the electric field is given by $\mathbf{E} = \frac{kr^2}{4\varepsilon_0}\hat{\mathbf{r}}$ (in spherical coordinate) for some constant *k*. (4 marks)

Solution:

From the Gauss law (differential form) we have:

$$\vec{\nabla}\mathbf{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = \varepsilon_0 \vec{\nabla}\mathbf{E}$$

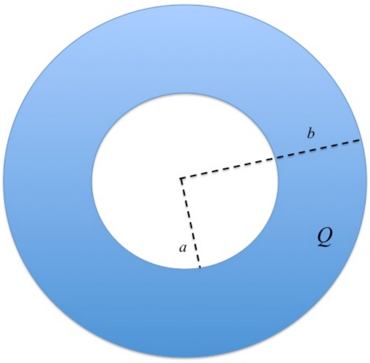
In spherical coordinates we know that:

$$\vec{\nabla} \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r} \frac{\partial E_\varphi}{\partial \varphi} + \frac{\partial E_z}{\partial z} \Rightarrow$$

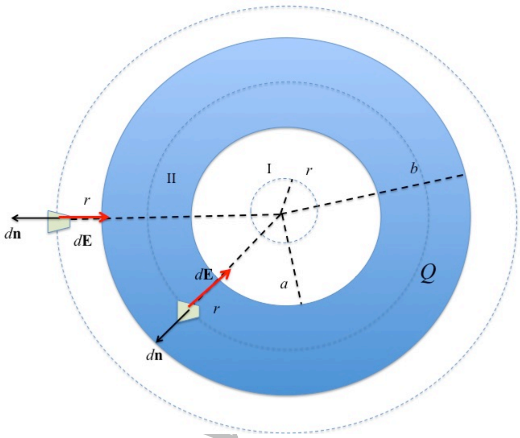
$$\vec{\nabla} \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{kr^2}{4\varepsilon_0}\right) = \frac{k}{4\varepsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^4\right) = \frac{k}{\varepsilon_0} r$$

$$\rho = \varepsilon_0 \vec{\nabla}\mathbf{E} = \varepsilon_0 \frac{k}{\varepsilon_0} r = kr$$

5. A charge Q (negative) is distributed uniformly in the shaded area between $r = a$ and $r = b$ inside a sphere shown in the figure. Find the electric field in all regions of space. (4 marks)



Solution:



We have to apply Gauss' Law for the three regions shown in the figure.

Region I: $0 < r < a$
Inside the Gaussian surface which is shown in the figure there is no charge. Thus

$$\int \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E} \cdot d\mathbf{A} = 0 \Rightarrow \mathbf{E} = 0$$

Region II: $a < r < b$
The first step we have to do is to find the charge density in this region:

$$\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3} = \frac{3}{4\pi}\frac{Q}{\left(b^3 - a^3\right)}$$

Inside the Gaussian surface II which is shown in the figure with radius r there is the following amount of charge:

$$q_{enc} = \rho V_{\mathrm{II}} = \frac{3}{4\pi}\frac{Q}{\left(b^3 - a^3\right)}\left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi a^3\right) = Q\frac{\left(r^3 - a^3\right)}{\left(b^3 - a^3\right)}$$

Thus applying the Gauss' Law we have

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int E\, dA = \frac{Q}{\varepsilon_0}\frac{\left(r^3 - a^3\right)}{\left(b^3 - a^3\right)} \Rightarrow$$

$$E\int dA = \frac{Q}{\varepsilon_0}\frac{\left(r^3 - a^3\right)}{\left(b^3 - a^3\right)} \Rightarrow E4\pi r^2 = \frac{Q}{\varepsilon_0}\frac{\left(r^3 - a^3\right)}{\left(b^3 - a^3\right)} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0\left(b^3 - a^3\right)}\frac{\left(r^3 - a^3\right)}{r^2}.$$

With the direction towards the center because the charge is negative.

Commented [VL1]: It is important mistake if you do not have an answer for the direction

Region III: $r > b$

Applying the Gauss' Law we have

$$\int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int E\, dA = \frac{Q}{\varepsilon_0} \Rightarrow$$

$$E\int dA = \frac{Q}{\varepsilon_0} \Rightarrow E4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi r^2}.$$

With the direction towards the center because the charge is negative.