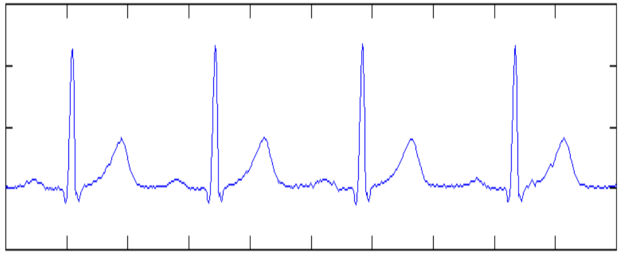
Chapter 1

Introduction to Signals and Systems

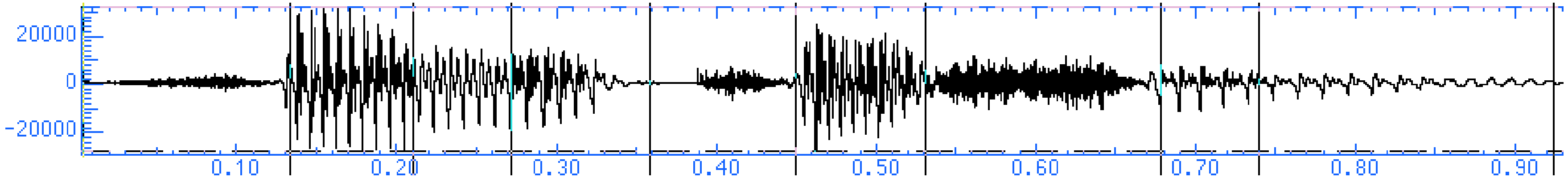
Signals and Systems

- The concept and theory of signals and systems are needed in almost all engineering and scientific disciplines.

Examples of 1-D Signals



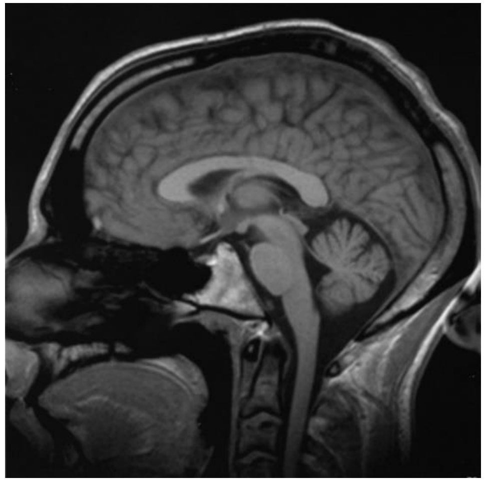
ECG signal



Speech Signal (oscillogram)

Examples of 2-D Signals

Gray-level Image

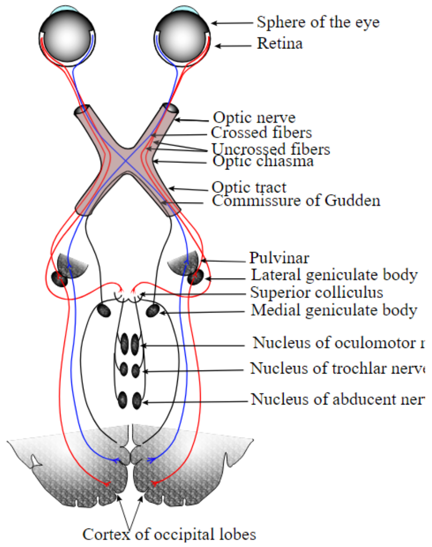


Biomedical Image

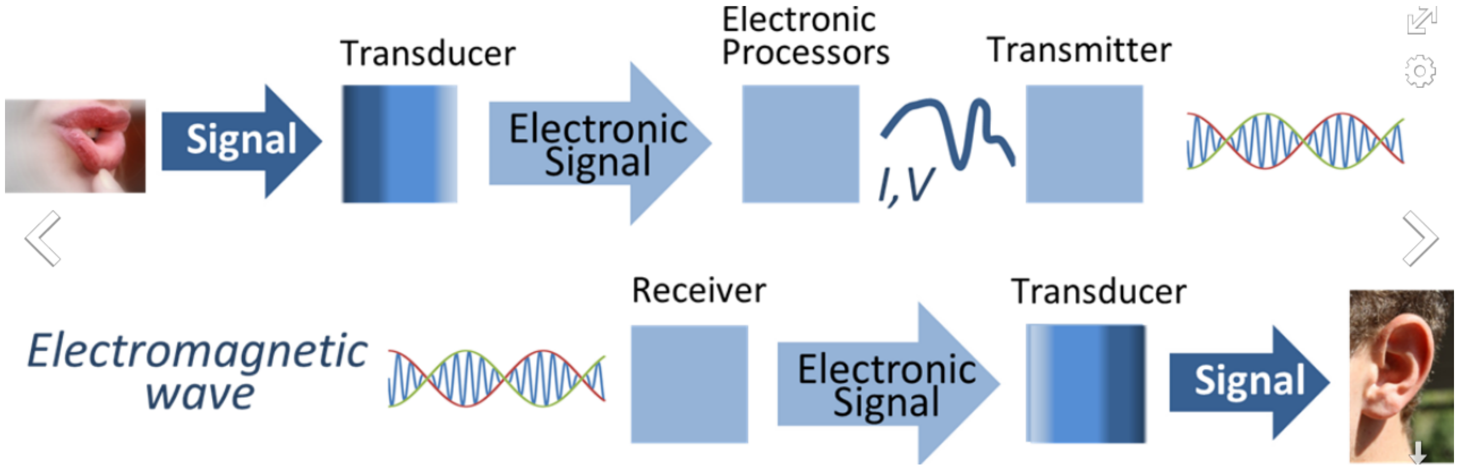
Color Image

- Intensity of the image at location (x, y) can be expressed as $I(x, t)$. Two independent variables (x and y), the image is a two dimensional signal.

Signals and Systems2



Sensory nervous system

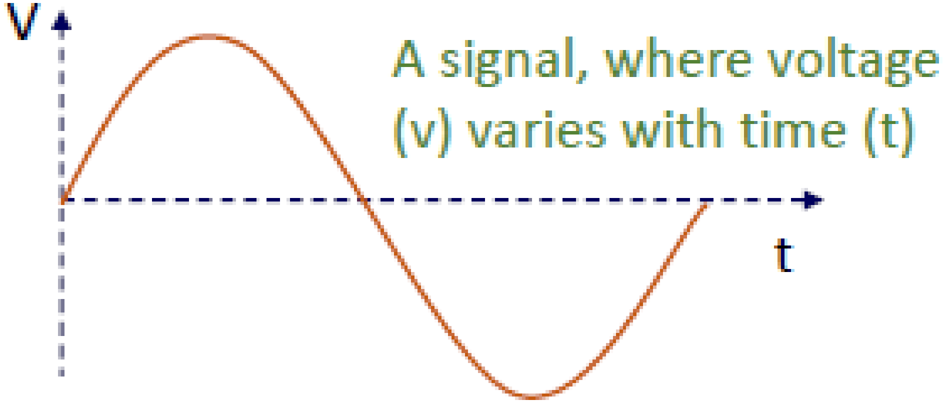


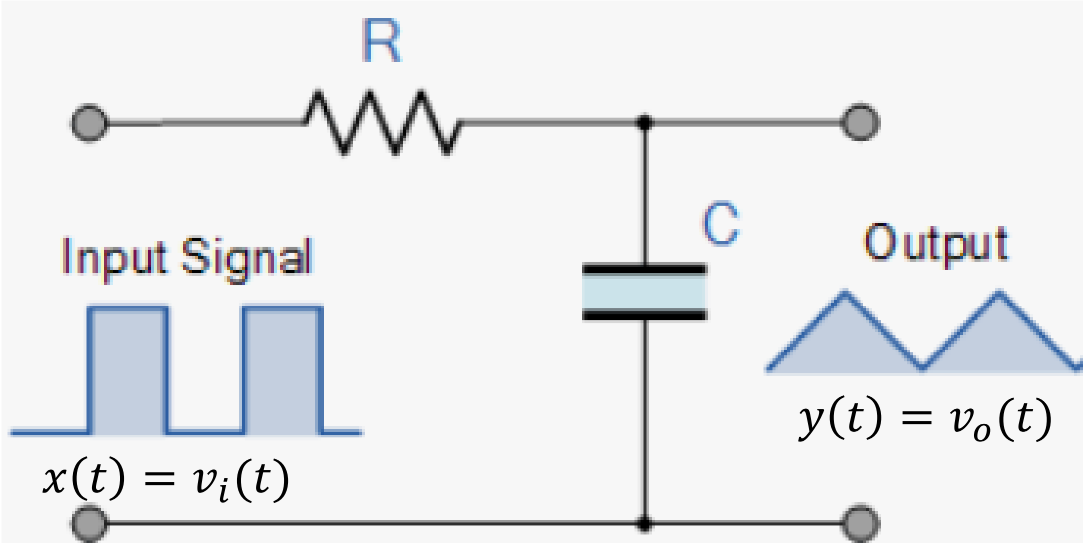
An electronic communications system

SIGNALS

- A *signal* is a function representing a physical quantity or variable (information about the behavior or nature of the phenomenon). Signals may describe a wide variety of physical phenomena.

- Mathematically, a signal (dependent variable) is represented as a function of an independent variable t (time) denoted $x(t)$

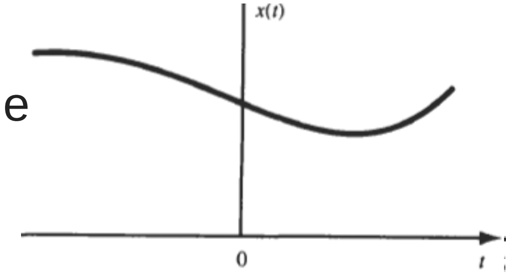




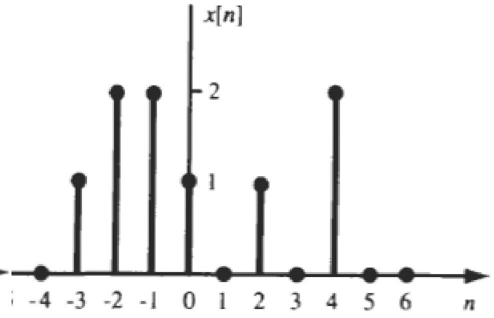
CLASSIFICATION OF SIGNALS

1. Continuous-Time and Discrete-Time Signals

A *continuous-time* (CT) signal is one that is present at all instants in time or space.
A *discrete-time* (DT) signal is only present at discrete points in time or space.



Continuous-Time Signal



Discrete-Time Signal

2. Analog and Digital Signals:

3. Real and Complex Signals: $x(t) = x_1(t) + j\, x_2(t)$ where $x_1(t)$ and $x_2(t)$ are ***real signals*** and $j = \sqrt{-1}$

4. Deterministic and Random Signals:

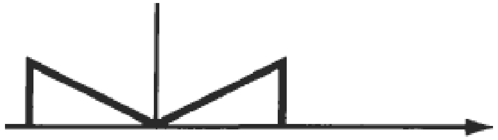
- ***Deterministic signals*** have values completely specified for any given time and can modeled by a known function of time.
- ***Random signals*** take random values at any given time and must be characterized statistically.

5. Even and Odd Signals:

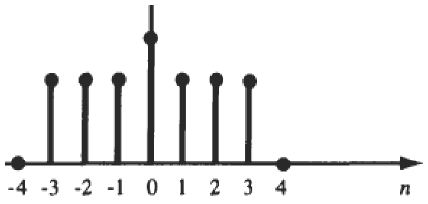
$x(t)$ is *even signal* if $x(-t) = x(t)$ $(x[-n] = x[n])$

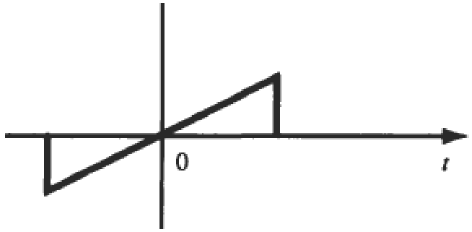
even signal is identical to its time-reversed counterpart about the origin

$x(t)$ is *odd signal* if $x(-t) = -x(t)$ $(x[-n] = -x[n])$

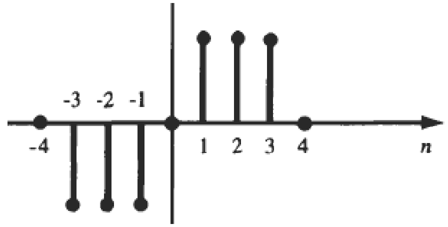


even signals





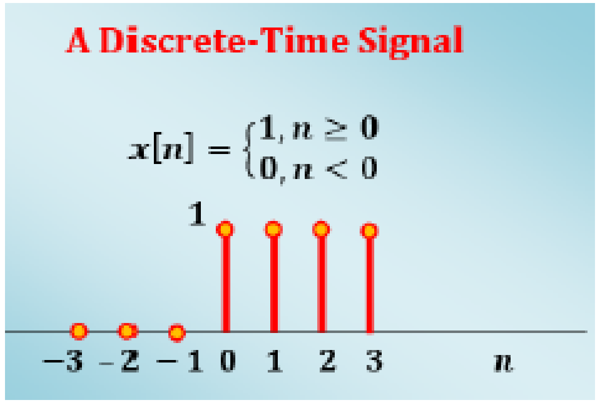
odd signals



$$x(t) = x_e(t) + x_o(t) \quad \text{with} \quad x_e(t) = \frac{1}{2}[x(t) + x(-t)] \text{ and } x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

CLASSIFICATION OF SIGNALS

Example: Decomposition a signal into Even and Odd parts



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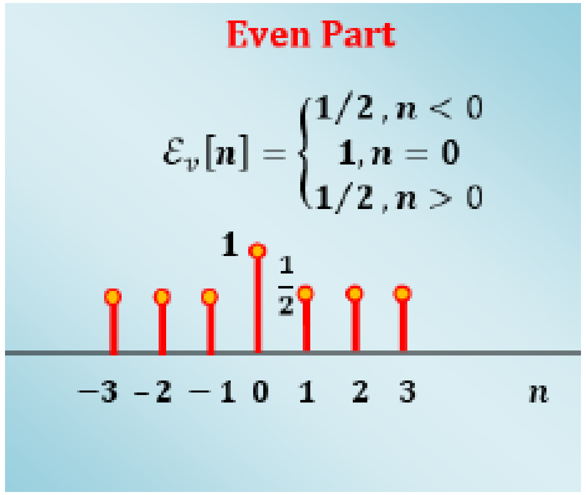

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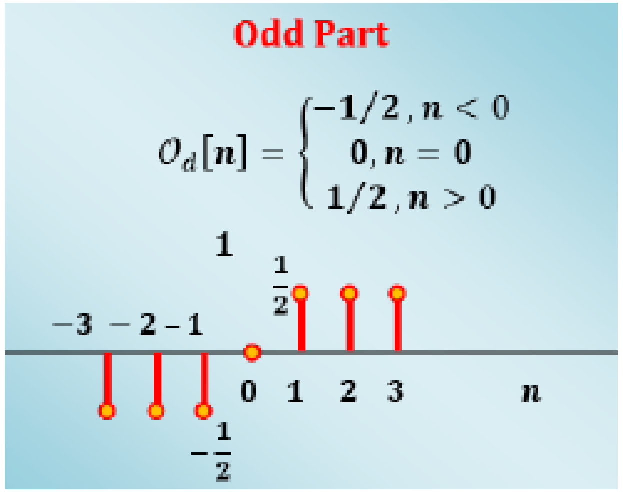
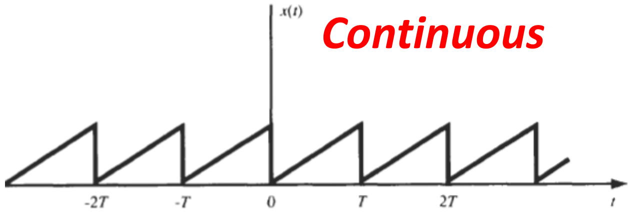


CLASSIFICATION OF SIGNALS2

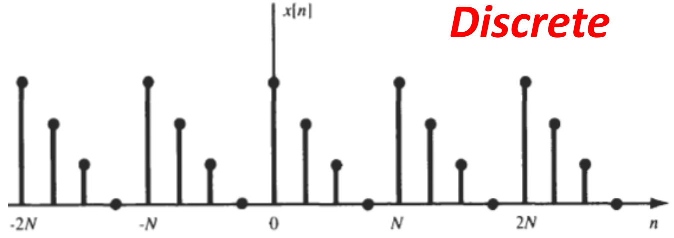
6. Periodic and Non-periodic Signals:

$\boldsymbol{x(t)}$ ($\boldsymbol{x[n]}$) is ***periodic*** with ***period*** T (N) if

The ***fundamental period*** T, of is the smallest positive value of T such that $\boldsymbol{x(t+mT)=x(t)}$



$$\boldsymbol{x(t+T)=x(t)} \quad \boldsymbol{T>0}$$



$$\boldsymbol{x[n+N]=x[n]}$$ N integer

Examples: Determine whether or not each of the following signals is periodic:

$$x(t)=2\cos\left(2t+{}^{\pi}/_{5}\right)$$

This signal is a CT sinusoid so it is periodic. Its fundamental angular frequency is 2 rad/sec and hence its fundamental period is

$$T=\frac{2\pi}{\omega_0}=\frac{2\pi}{2}=\pi$$

$$x[n]=\cos(3n)$$

$$x[n+N_0]=\cos(3[n+N_0])=\cos(3n+3N_0)=\cos(3n+2m\pi)$$

This suggests that $3N_0=2m\pi \rightarrow N_0=\dfrac{2}{3}m\pi$

Since π is irrational $\nexists\, m \in \mathbb{Z}\ s.t.\ \frac{2}{3}m\pi \in \mathbb{Z}^+$

therefore $x[n]$ is not periodic.

CLASSIFICATION OF SIGNALS3

7. Energy and Power Signals:

Continuous

Total Energy E_∞ of $\boldsymbol{x(t)}$: $E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt \;\; (joules)$

Total averaged Power P_∞ of $\boldsymbol{x(t)}$: $P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \;\; (watts)$

$\boldsymbol{x(t)}$ is an ***energy signal*** if $0 < E_\infty < \infty$ then $P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$ Signals with finite total energy

$\boldsymbol{x(t)}$ is an ***power signal*** if $0 < P_\infty < \infty$ then $E_\infty = \infty$ Signals with finite average power

$\boldsymbol{x(t)}$ can be with neither E_∞ nor P_∞ finite.

Discrete

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Example: $x(t) = e^{-2t}u(t)$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |e^{-2t}u(t)\,|^2 dt = \int_{0}^{\infty} e^{-4t} dt = \frac{1}{-4} e^{-4t}\Big|_0^\infty = \frac{1}{4}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{-8T} e^{-4t}\Big|_0^T = \lim_{T\to\infty} \frac{1}{-8T} [e^{-4T} - e^0] = 0$$

or $P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$

P_∞ is zero, because $E_\infty < \infty$

Example: $x[n] = \cos\left(\frac{\pi}{4} n\right)$

$cos(2\alpha) = cos^2(\alpha) - sin^2(\alpha)$

$1 = cos^2(\alpha) + sin^2(\alpha)$

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} \left|\cos\left(\frac{\pi}{4} n\right)\right|^2 = \sum_{n=-\infty}^{\infty} \cos^2\left(\frac{\pi}{4} n\right) = \sum_{n=-\infty}^{\infty} \frac{1 + cos\left(\frac{\pi}{2} n\right)}{2}$$

$$E_\infty = \sum_{n=-\infty}^{\infty} \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{cos\left(\frac{\pi}{2} n\right)}{2} = \infty$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1 + cos\left(\frac{\pi}{2} n\right)}{2}$$

Sum of cos on a period is zero

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \left[\sum_{n=-N}^{N} \frac{1}{2} + \sum_{n=-N}^{N} \frac{cos\left(\frac{\pi}{2} n\right)}{2} \right] = \frac{1}{2}$$

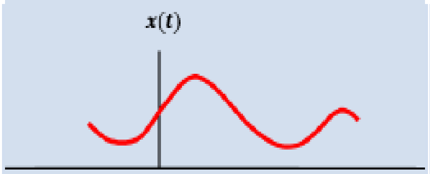
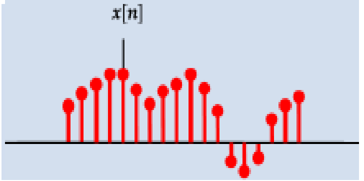
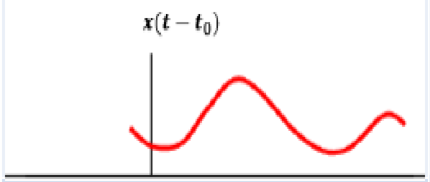
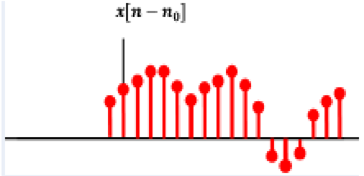
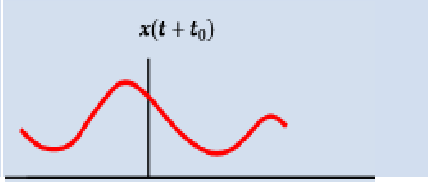
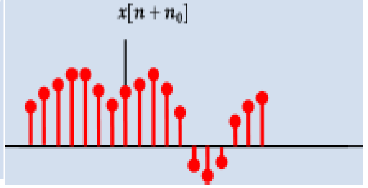
$$\sum_{n=-N}^{N} \frac{1}{2} = \frac{2N+1}{2}$$

Transformation of the independent Variable

A general form of transformation of independent variable is $x(\beta t + \propto)$, where $\propto$ and β are given numbers.

1 Time shift ($\beta = 1, \propto \neq 0$)

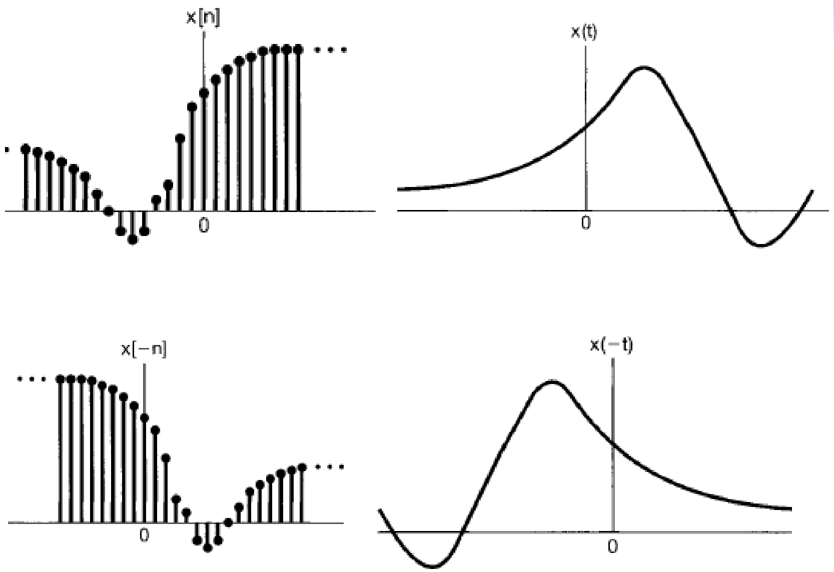
The original and the shifted signals are identical in shape, but that are displaced or shifted relative to each other (delayed or advanced).

Signals	Continuous time	Discrete time
Original		
Delayed		
Advance		

Such signals arise in applications such radar, sonar and seismic signal processing. Shifted signal due to the transmission time.

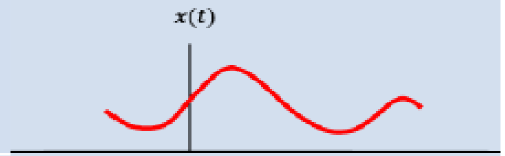
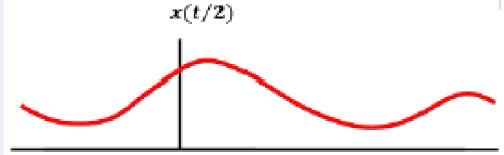
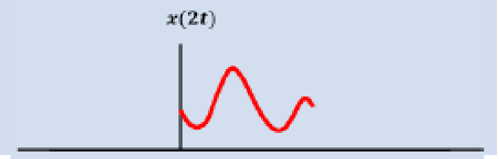
2 Time reversal (Reflection) $\beta = -1$

The reflected signal $x(-t)$ or $x[-n]$ is obtained from the signal $x(t)$ or $x[n]$by a reflection about $t = 0$ or $n = 0$



3 Time Scaling

The time-scaled signal $x(\beta t)$ is obtained from the signal $x(t)$ by multiplying the time variable by a constant β

Signals	Continuous time
Original	
Stretching	
Compressing	

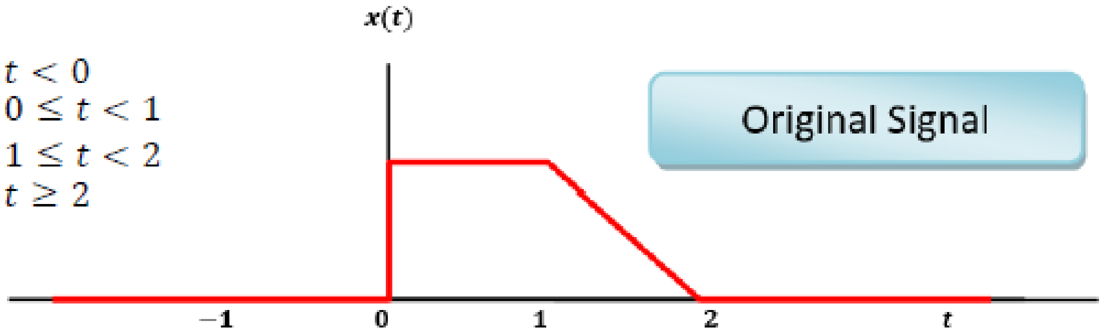
if $\beta > 1$: Compressing

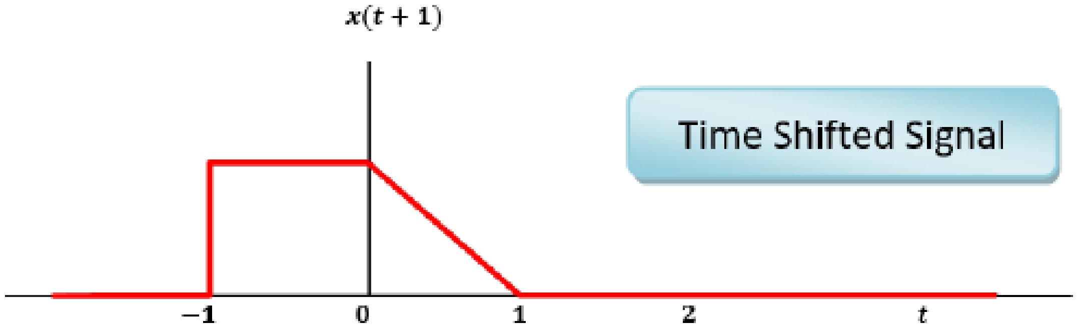
if $\beta < 1$: Stretching

Transformation of the independent Variable 2

Example1: Time Shift The signal $x(t+1)$ can be obtained by shifting $x(t)$ to the left by one unit

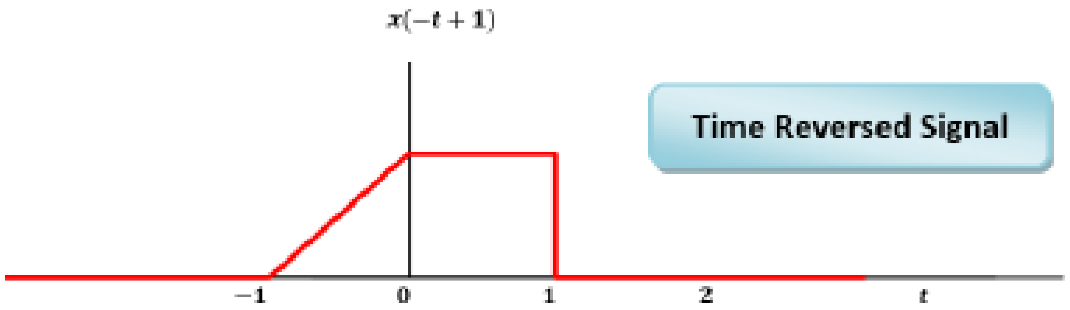
$$x(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ 0 \le t < 1 \\ 2-t & if\ 1 \le t < 2 \\ 0 & if\ t \ge 2 \end{cases}$$





Example2: Time reversal The signal $x(-t+1)$ can be obtained from $x(t)$ using the mathematical definition

t	$-t+1$	$x(-t+1)$
-2	3.0	0
-1.5	2.5	0
-1	2.0	0
-0.5	1.5	0.5
0	1.0	1
0.5	0.5	1
1	0.0	1
1.5	-0.5	0
2	-1.0	0
2.5	-1.5	0
3	-2.0	0

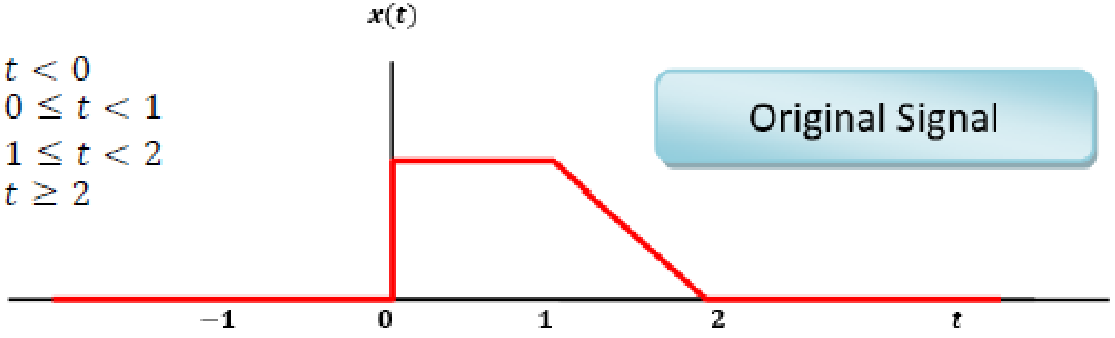


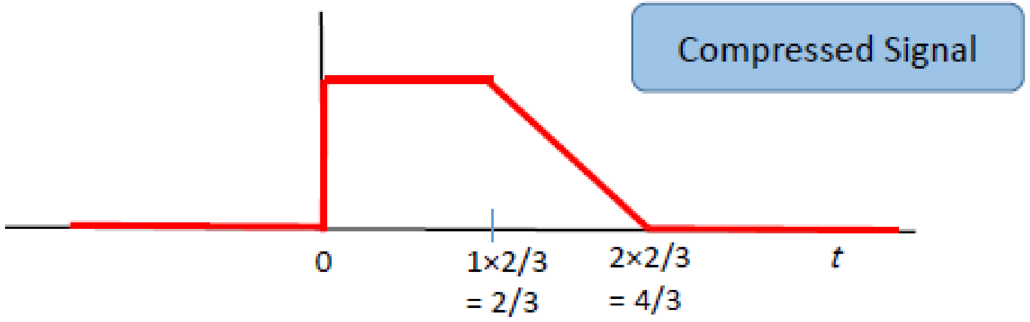
First plot $x(t+1)$, then reflect.

Transformation of the independent Variable3

Example3: Time Compression Find $x(\frac{3}{2}t)$ ($|\beta| = \frac{3}{2} > 1$ linear compression by a factor of $\frac{1}{(^3/_2)} = \frac{2}{3}$)

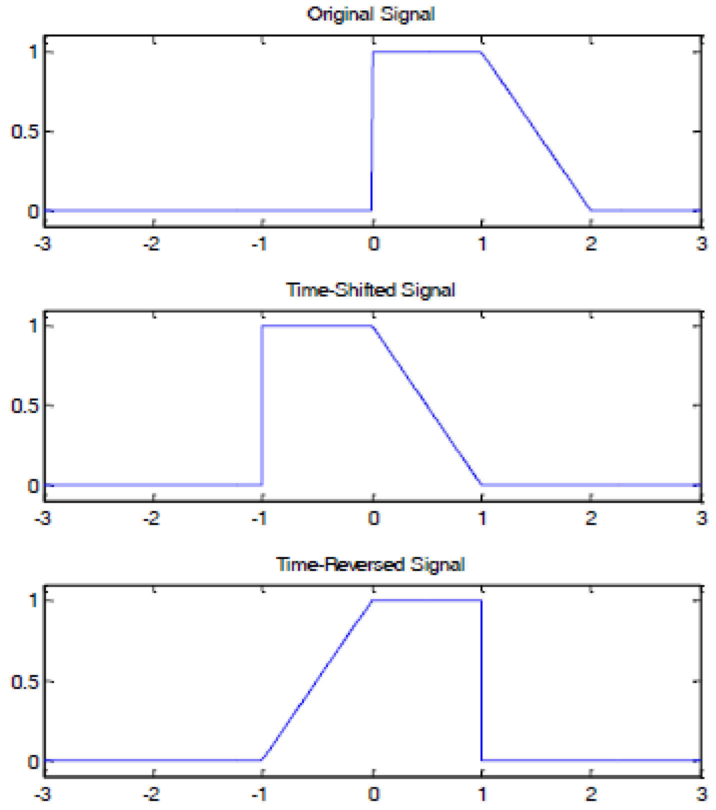
$$x(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ 0 \le t < 1 \\ 2-t & if\ 1 \le t < 2 \\ 0 & if\ t \ge 2 \end{cases}$$





Matlab

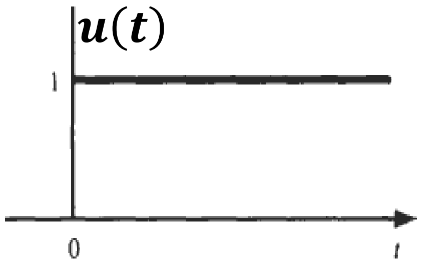
```
>>g = inline(' ((t>=0)&(t<1)) + (2-t).*((t>=1) & (t<2))','t');
>>t = -3:0.001:3;
>>subplot(3,1,1),  plot(t,  g(t)),  axis([-3  3  -0.1  1.1]),
title('Original Signal')
>>subplot(3,1,2),  plot(t,  g(t+1)),  axis([-3  3  -0.1  1.1]),
title('Time-Shifted Signal')
>>subplot(3,1,3),plot(t,  g(-t+1)),axis([-3  3  -0.1  1.1]),
title('Time-Reversed Signal')
```



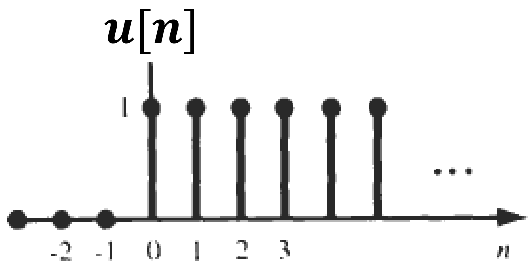
BASIC SIGNALS

1. The Unit Step signal (Heaviside unit function):

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



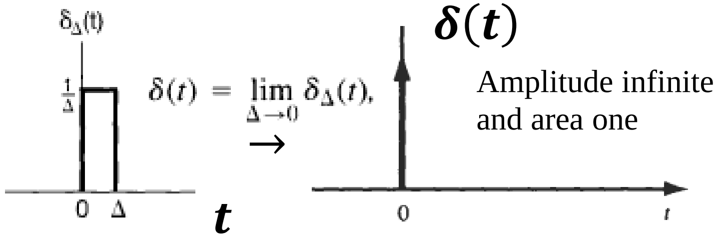
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



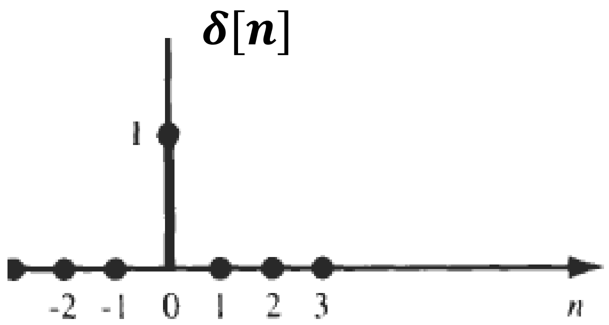
2. The Unit Impulse signal (Dirac delta function):

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1$$



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



properties of $\delta(t)$
$$\begin{cases} \delta(at) = \dfrac{1}{|a|}\delta(t) \\ \delta(-t) = \delta(t) \\ x(t)\delta(t) = x(0)\delta(t) \end{cases}$$

$$\int_{-\infty}^{t} \delta(t)\, dt = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} = u(t) \rightarrow \delta(t) = \frac{du(t)}{dt}$$

$$x[n]\delta[n] = x[0]\delta[n]$$

$$x[n]\delta[n-k] = x[k]\delta[n]$$

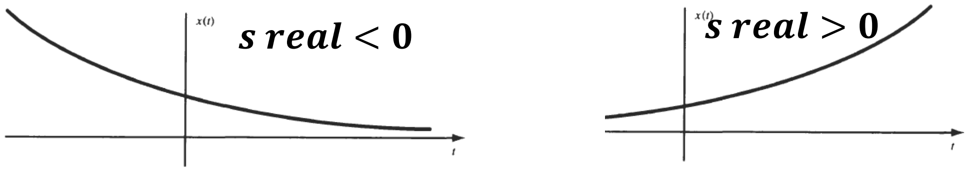
$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

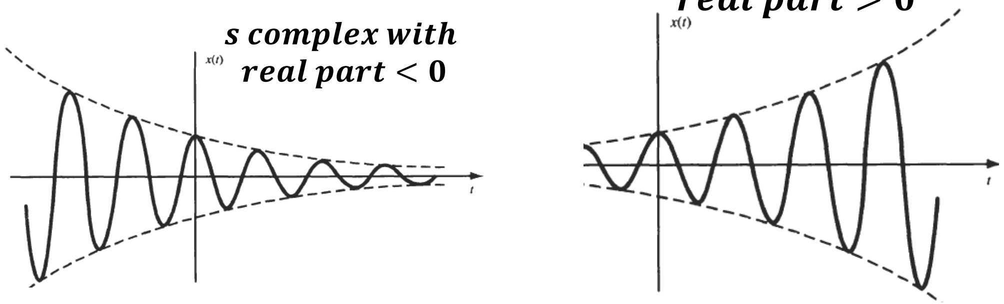
any sequence $x[n]$ can be expressed

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

BASIC SIGNALS 3



$$x(t) = C\,e^{st}$$



3. Complex Exponential Signals:

Using Euler's formula ($e^{j\theta} = \cos(\theta) + j\sin(\theta)$)

$x(t) = e^{jw_o t} = \cos(w_o t) + j\sin(w_o t)$ is a complex signal

$x(t)$ is periodic with fundamental period $T_0 = \frac{2\pi}{w_0}$

Complex Exponential Sequences

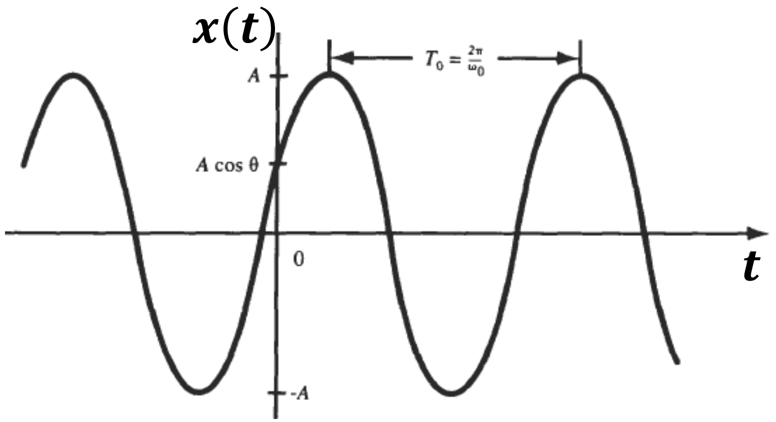
$$x[n] = e^{j\Omega_o n} = \cos(\Omega_o n) + j\sin(\Omega_o n)$$

Period $N = m\frac{2\pi}{\Omega_o}$ (m: Positive integer)

4. Sinusoidal Signals:

$$x(t) = A\cos(w_o t + \theta)$$

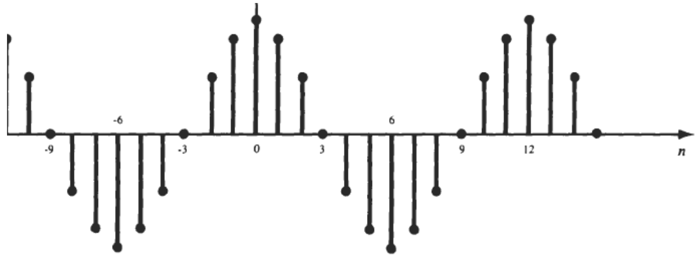
(Amplitude: A; Angular Frequency (pulsation): w_o; time: t; Phase: θ)



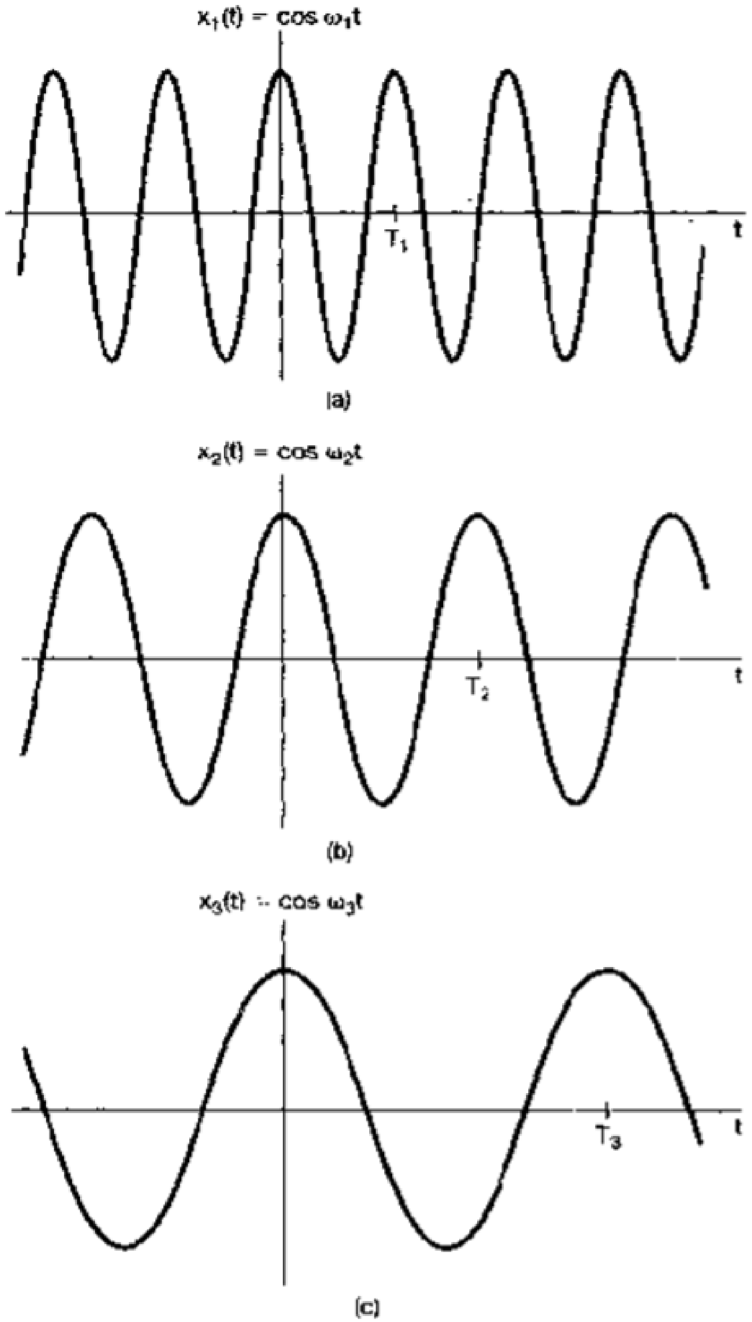
$x(t)$ has a fundamental

- *period* $T_0 = \frac{2\pi}{w_0}$
- *frequency* $f_0 = \frac{1}{T_0}$
- *angular frequency* $w_0 = 2\pi f_0$

Sinusoidal Sequences: $x[n] = A\cos(\Omega_o n + \theta)$



Fundamental Period and Frequency



$$x(t) = A\,cos(w_o t + \theta) \quad \text{with} \quad w_o = 2\pi F_0 = \frac{2\pi}{T_0}$$

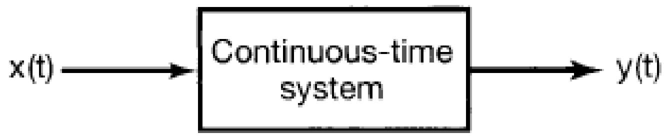
If we decrease the value of the magnitude of ω_0, we slow down the rate of oscillations and hence increase the period T_0. Alternatively, if we increase the value of the magnitude of ω_0, we increase the rate of oscillations and hence decrease the period T_0.

$$\omega_1 > \omega_2 > \omega_3$$
$$T_1 < T_2 < T_3$$

SYSTEMS

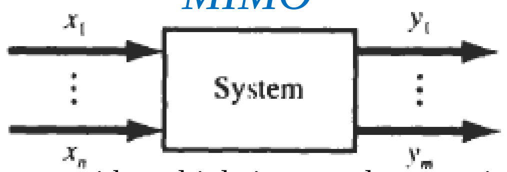
- A *system* is a mathematical model of a physical process (an interconnection of components, devices, or subsystems) that transforms an *input signal* (excitation, single or multiple) into an *output signal* (response, single or multiple).

SISO



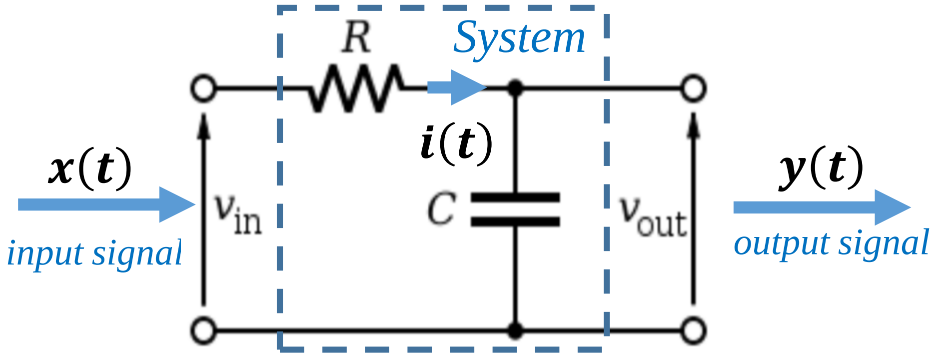
System with single input and output signals

MIMO



System with multiple input and output signals

Example 1: RC circuit



$$y(t) = v_c(t) = \frac{1}{C}\int i(t)dt \rightarrow i(t) = C\frac{dv_c(t)}{dt}$$

$$i(t) = \frac{v_R(t)}{R} = \frac{x(t) - y(t)}{R} = C\frac{dy(t)}{dt}$$

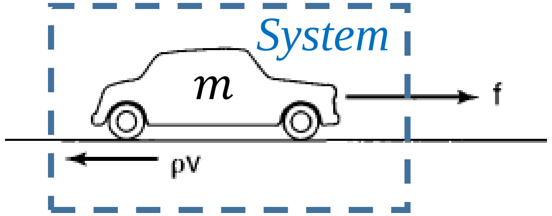
$$\rightarrow \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

differential equation describing the relationship between $x(t)$ and $y(t)$

Example 2: An automobile

An automobile with mass m responding to an applied force $f(t)$ (*input*) from the engine and to a retarding frictional force $\rho v(t)$ proportional to the automobile's velocity $v(t)$(*output*).

$$\frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)]$$



$$\rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\rightarrow \frac{dy(t)}{dt} + \frac{\rho}{m}y(t) = \frac{1}{m}x(t)$$

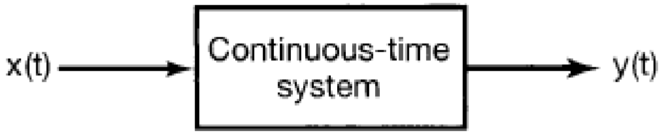
differential equation describing the relationship between $f(t)$ and $v(t)$

CLASSIFICATION OF SYSTEMS

1. Continuous-Time and Discrete-Time Systems:

If the input and output signals are continuous-time signals, then the system is called a ***continuous-time system.*** If the input and output signals are discrete-time signals or sequences, then the system is called a ***discrete-time system.***

Discrete System with single input and output signals



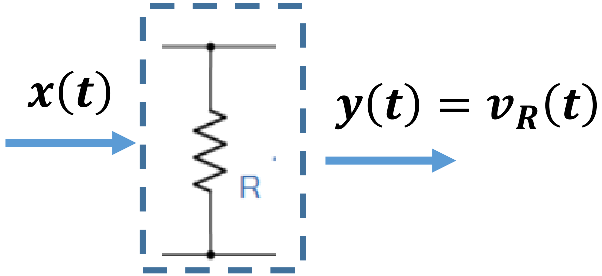
Continuous System with single input and output signals

2. Systems with Memory and without Memory

A system is ***memoryless*** if its output at any time depends only at that same time. Otherwise, the system is said to ***have memory***.

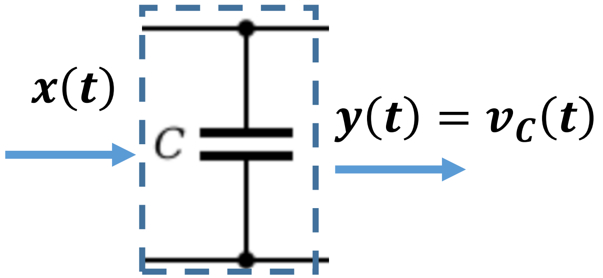
Memoryless system: A resistor R with a current as input $\boldsymbol{x(t)}$and a voltage as output $\boldsymbol{y(t)}$

$$y[n] = 2\,x[n] - x^2[n]$$



$$y(t) = R\,x(t)$$

system with memory: a capacitor $\boldsymbol{C}$ with the current as input $\boldsymbol{x(t)}$ and voltage as output $\boldsymbol{y(t)}$



$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau$$

the accumulator $$y[n] = \sum_{k=-\infty}^{n} x[k] = y[n-1] + x[n]$$

3. Causal and Non-causal Systems:

A system is *causal* if the output at any time depends only on *values of the input at the present time and in the past* (non-anticipative of future values of the input). All memoryless systems are causal, but not vice versa.

Non-causal systems

$$y(t) = x(t+1) \qquad y[n] = x[n+2] \qquad y[n] = x[-n]$$

causal systems

$$y(t) = x(t) + x(t-1)$$

the current value of the input x(t) influences the current value of the output y(t)

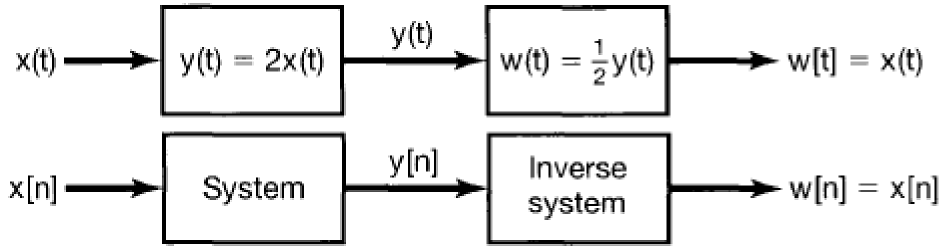
$$y(t) = x(t)\cdot \cos(t+1)$$

$$y[n] = x[n-2]$$

CLASSIFICATION OF SYSTEMS2

4. Invertibility and Inverse Systems:

A system is said to be *invertible* if distinct inputs lead to distinct outputs.



Examples:

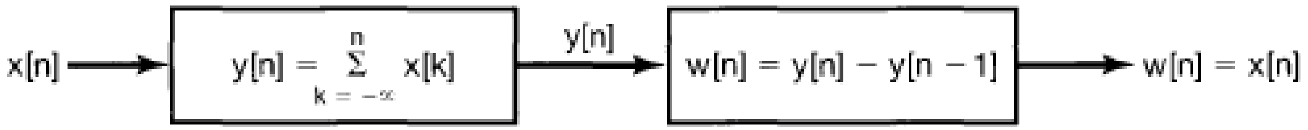
- an *invertible* continuous-time system is $y(t) = 2\,x(t)$

$$y(t) = 2\,x(t) \rightarrow w(t) = x(t) = 0.5\,y(t)$$

- a *Non-invertible* continuous-time system is $y(t) = x^2(t)$

we cannot determine the sign of the input from knowledge of the output.

- the accumulator (*invertible*)

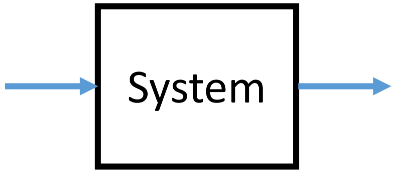


- *Non-invertible* $y(t) = 0$ For different inputs $x(t)$ the output $y(t)$ is zero

5. Linear Systems and Nonlinear Systems:

A system is *linear if it* possesses the property of superposition (Homogeneity and additivity)

$x_1(t)$
$x_2(t)$
$a\,x_1(t) + bx_2(t)$



$y_1(t)$
$y_2(t)$
$a\,y_1(t) + by_2(t)$

Examples:

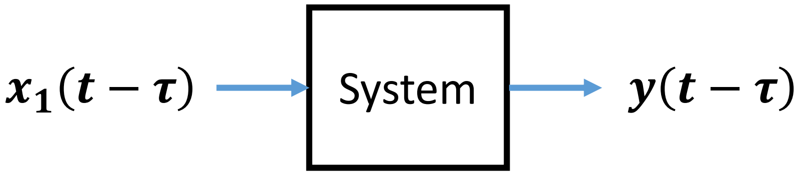
$y(t) = x^2(t)$ *is a nonlinear system*

$y(t) = a\,x(t)$ *is a linear system*

CLASSIFICATION OF SYSTEMS3

6. Time-Invariant and Time-Varying Systems:

A system is called *time-invariant* if a time shift (delay or advance) in the input signal causes the same time shift in the output signal (behavior and characteristics of the system are fixed over time).



If the system is *linear* and also *time-invariant*, then it is called a linear rime-invariant system (**LTI system**).

Examples: For the system: $y(t) = sin[x(t)]$

For input $x_1(t)$: $y_1(t) = sin[x_1(t)]$

For input $x_2(t) = x_1(t-t_0)$

$$y_2(t) = sin[x_2(t)] = sin[x_1(t-t_0)]$$

Delaying: $y_1(t)$: $y_1(t-t_0) = sin[x_1(t-t_0)] = y_2(t)$

system is time invariant.

7. Stability

A system is *bounded-input/bounded-outpu*t (***BIBO***) stable if for any bounded input x ($|x| \leq k_1$) the corresponding output y is also bounded ($|y| \leq k_2$). A stable system is one in which small inputs lead to responses that do not diverge.

Examples:

$$y(t) = t\, x(t)$$

For bounded input ($x(t) = 1$) the output $y(t) = t$ unbounded.

unstable.

$$y(t) = e^{x(t)}$$

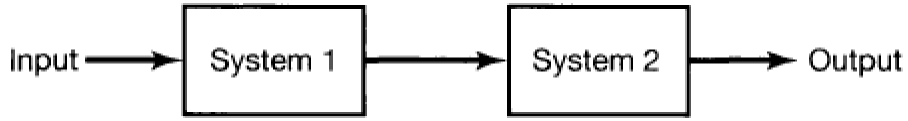
For bounded input $|x(t)| < B$ the output $e^{-B} < |y(t)| < e^{B}$ bounded.

stable.

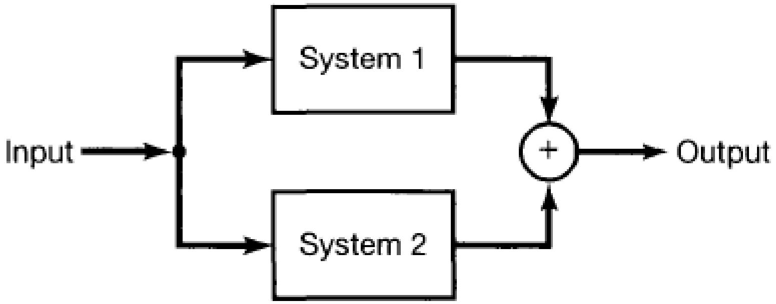
Interconnections of Systems

- Many real systems are built as interconnections of several subsystems.

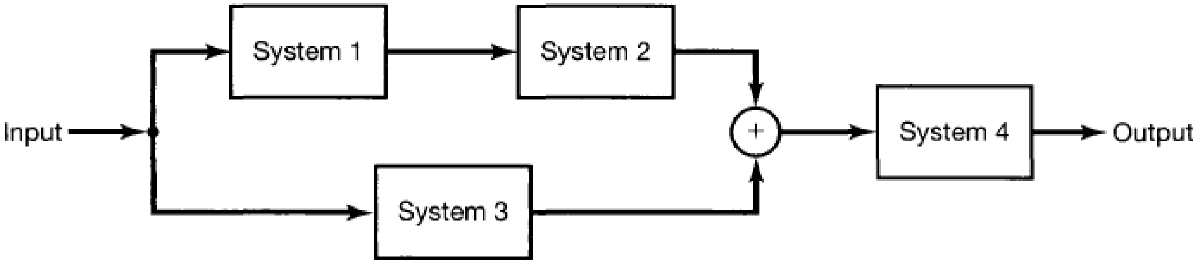
Interconnection of two systems



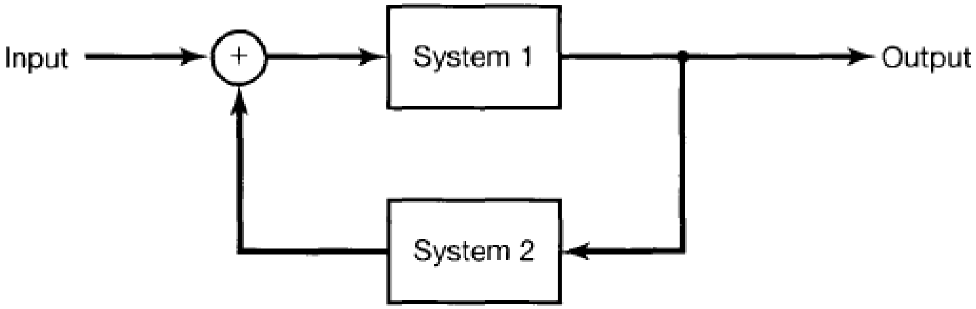
series (cascade) interconnection



parallel interconnection



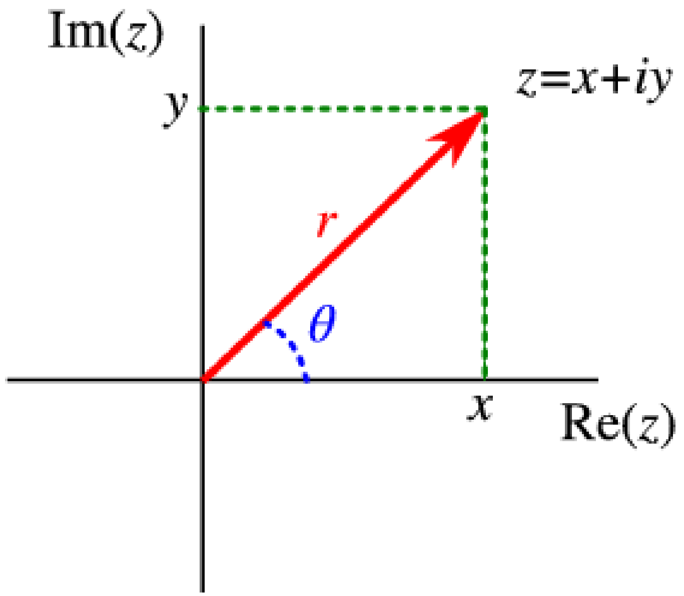
series-parallel interconnection



Feedback interconnection

Background on complex numbers

- Cartesian (rectangular) Form: $z = x + jy$ $\quad j = \sqrt{-1}$



- Polar form: $z = r\, e^{j\theta}$
  - $r = |z| = \sqrt{x^2 + y^2}$ Magnitude of z
  - $\theta = \text{atan}(y/x)$ Phase (argument) of z

- Euler formula: $e^{j\theta} = \cos(\theta) + j sin(\theta)$

$$z = |z|(\cos(\theta) + j sin(\theta)\,)$$

- Polar form is more convenient for multiplication and divisions
- Cartesian form is more convenient for addition ad subtraction

Background on complex numbers2

Example:

- Express the following number in Cartesian form: $z = \sqrt{2}e^{j\pi/4}$ $z = 0.5e^{-j\pi}$

$$z = \sqrt{2}e^{j\pi/4} = |z|e^{j\theta} = \sqrt{2}\left(\cos(\pi/4) + j\sin(\pi/4)\right) = 1 + j$$

$$z = 0.5e^{-j\pi} = |z|e^{j\theta} = 0.5\left(\cos(-\pi) + j\sin(-\pi)\right) = -0.5$$

- Express the following number in polar form: $z = 5$ $z = 1 + j$

$$z = 5 = \sqrt{5^2 + 0^2}e^{j\mathrm{atan}(0/5)} = 5e^{j0} = 5(\cos(0) + j\sin(0))$$

$$z = 1 + j = \sqrt{1^2 + 1^2}e^{j\mathrm{atan}(1/1)} = \sqrt{2}e^{j\pi/4} = \sqrt{2}\left(\cos(\pi/4) + j\sin(\pi/4)\right)$$