Hindawi
Journal of Function Spaces
Volume 2019, Article ID 1691903, 12 pages
https://doi.org/10.1155/2019/1691903

Research Article

Directional Multifractal Analysis in the L^p Setting

Mourad Ben Slimane ,[1] Ines Ben Omrane,[2] Moez Ben Abid,[3] Borhen Halouani,[1] and Farouq Alshormani[1]

[1] *King Saud University, Department of Mathematics, College of Science, P. O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 90950, Riyadh 11623, Saudi Arabia*
[3] *Université de Sousse, Ecole Supérieure des Sciences et de la Technologie de Hammam Sousse, Sousse, Tunisia*

Correspondence should be addressed to Mourad Ben Slimane; mbenslimane@ksu.edu.sa

Received 11 March 2019; Accepted 13 June 2019; Published 16 July 2019

Academic Editor: Alberto Fiorenza

The classical Hölder regularity is restricted to locally bounded functions and takes only positive values. The local L^p regularity covers unbounded functions and negative values. Nevertheless, it has the same apparent regularity in all directions. In the present work, we study a recent notion of directional local L^p regularity introduced by Jaffard. We provide its characterization by a supremum of a wide range oriented anisotropic Triebel wavelet coefficients and leaders. In addition, we deduce estimates on the Hausdorff dimension of the set of points where the directional local L^p regularity does not exceed a given value. The obtained results are illustrated by some examples of self-affine cascade functions.

1. Introduction

Multifractal analysis describes geometrically and statistically the distribution of pointwise regularities (or singularities) of irregular functions f on $\mathbb{R}^m$. It was first introduced in the context of the statistical study of fully developed turbulence in the mid 80's [1]. The classical notion of pointwise regularity most commonly used is Hölder regularity. Let $y \in \mathbb{R}^m$ and $\alpha > 0$. Let $B(y,r)$ denote the open ball of radius r, centered at y. Recall that a function f on $\mathbb{R}^m$ belongs to $C^\alpha(y)$ if there exist constants C and $0 < R < 1$ and a polynomial P of degree at most the integer part $[\alpha]$ of α, such that

$$\forall x \in B(y,R) \quad \left| f(x) - P(x-y) \right| \le C \left| x-y \right|^\alpha . \tag{1}$$

Clearly, this condition makes sense only if f is locally bounded. It is equivalent to

$$\forall r \le R \quad \left\| f(\cdot) - P(\cdot - y) \right\|_{L^\infty(B(y,r))} \le C r^\alpha . \tag{2}$$

It describes how the L^∞ norm of f, properly renormalized by substracting a polynomial, behaves in small neighborhoods of y.

The Hölder exponent of f at y is defined as

$$\alpha(y) = \sup \{\alpha;\ f \in C^\alpha(y)\} . \tag{3}$$

The multifractal formalism relates some function spaces norms of f to the Hölder spectrum (which is the Hausdorff dimension of the set of points where f has a given Hölder exponent). The idea of using wavelets in multifractal analysis has been worked out first by Arneodo et al. [2]. Wavelets characterize Hölder regularity and many function spaces.

In many PDE's, the natural function space setting is L^p for $1 \le p < \infty$ or a Sobolev space which includes unbounded functions. In such cases, one replaces the L^∞ norm in (2) by the L^p norm. This corresponds to the following weaker condition introduced by Calderón and Zygmund [3].

Definition 1 (let $p \ge 1$). Let f be a function which belongs locally to $L^p(\mathbb{R}^m)$.

If $\alpha \geq 0$, one says that $f \in T_\alpha^p(y)$ if there exist constants C and $0 < R < 1$ and a polynomial P of degree at most $[\alpha]$ such that

$$\forall r \leq R \quad \|f(\cdot) - P(\cdot - y)\|_{L^p(B(y,r))} \leq Cr^{\alpha+m/p}. \tag{4}$$

If $-m/p \leq \alpha < 0$, one says that $f \in T_\alpha^p(y)$ if there exist constants C and $0 < R < 1$ such that

$$\forall r \leq R \quad \|f\|_{L^p(B(y,r))} \leq Cr^{\alpha+m/p}. \tag{5}$$

The p−exponent of f at y is defined as

$$\alpha^p(y) = \sup\{\alpha;\ f \in T_\alpha^p(y)\}. \tag{6}$$

Clearly if $f \in L_{loc}^p(\mathbb{R}^m)$ then $f \in T_{-m/p}^p(y)$. The usual Hölder regularity $C^\alpha(y)$, for $\alpha > 0$, corresponds to $p = \infty$. If $f \in C^\alpha(y)$, then $f \in T_\alpha^p(y)$ for all $p \geq 1$. If $f \in T_\alpha^p(y)$ then $f \in T_\alpha^{p'}(y)$ for all $p' \leq p$.

In [4, 5], Jaffard and Mélot proved that if Ω is a domain of $\mathbb{R}^m$ and y is on the boundary $\partial\Omega$ of Ω, then by taking $P = 0$ or $P = 1$, the $T_{\alpha/p}^p(y)$ condition for the characteristic function χ_Ω of Ω coincides with the bilaterally weak α accessibility of Ω at y. Let $\alpha \geq 0$. Recall that Ω is called bilaterally weak α accessible at y if there exist two constants C and $0 < R < 1$ such that

$$\forall r \leq R \quad \min\{meas(\Omega^c \cap B(y,r)), meas(\Omega \cap B(y,r))\} \leq Cr^{\alpha+m}, \tag{7}$$

where *meas* is the Lebesgue measure. This remark allows performing a multifractal analysis of fractal boundaries [5, 6]. This analysis has many applications in physics, mechanics, or chemistry where many phenomena involve fractal interfaces. More pointwise exponents for classification of the geometry of fractal boundaries were studied by Jaffard and Heurteaux [7]. The relationship of these exponents to local dimension computation was proved by Tricot [8].

In [4, 5], a wavelet characterization of the $T_\alpha^p(y)$ regularity was obtained. Moreover, an associated multifractal formalism was conjectured: the Hausdorff dimension $d_p(H)$ of the set of points y, where the p−exponent $\alpha^p(y)$ of a function f is equal to H, may be derived from some global functional norms extracted from f.

Unfortunately, if $m \geq 2$ then Definition 1 has the same apparent value in all coordinate directions. However many signals belong to classes of functions with various directional regularity behaviors (see, for instance, [9–17] and the references therein). These behaviors are important for detection of edges, efficient image compression, etc. (see, for instance, [12] and the references therein). Classical (isotropic) wavelets are not optimal for analyzing directional or anisotropic features like edges. Extensions of wavelet bases which can be elongated in particular directions were considered. Candes and Donoho [18] and Mallat [19] (resp., Donoho [20] and Guo and Labate [21]) have used ridgelets and bandelets (resp., wedgelets and shearlets) to detect singularities along lines and hyperplanes (resp., discontinuities along smooth edges). Sampo and Sumetkijakan [16, 22, 23] (resp., Jaffard [24]) have used curvelets and Hart-Smith transform (resp., the anisotropic Gabor-wavelet transform) to detect directional pointwise Hölder singularities.

Many signals present anisotropies quantified through regularity characteristics and features that strongly differ when measured in different directions [13, 14, 24–29]. Several authors were concerned with the problem of obtaining useful decompositions of isotropic and anisotropic function spaces in simple building blocks: atoms, quarks, wavelets, splines (see, for example, Farkas [30], Garrigós et al. [31, 32], Hochmuth [33], and Kamont [34]). Triebel family of anisotropic wavelets yield characterizations of anisotropic Besov spaces [35, 36] and anisotropic Hölder regularity (see Ben Slimane et al. [37–39]).

Let $e \in \mathbb{R}^m$ be a unit vector. To take into account pointwise directional L^p behavior in direction e, it is natural to define the local L^p regularity $\alpha^{p,e}(y)$ at a point y in direction e as the p−exponent at 0 of the one variable function $f_{y,e} : t \longmapsto f(y + te)$, that is,

$$\alpha^{p,e}(y) = \sup\left\{\alpha;\ f_{y,e} \in T_\alpha^p(0)\right\}. \tag{8}$$

We wish to derive or estimate the Hausdorff dimension $d_{p,e}(H)$ of the set of points y where $\alpha^{p,e}(y) = H$ from some global quantities extracted from the function f itself and not its traces $f_{y,e}$ (actually these traces are unknown since corresponding points y are unknown). One cannot expect directional local L^p regularity $\alpha^{p,e}(y)$ to be characterized in terms of the size of the wavelet coefficients by taking wavelets on $\mathbb{R}^m$ because $f_{y,e}$ is defined as the trace of f on a line, which is a set of vanishing measure and wavelets on $\mathbb{R}^m$ have a support of nonempty interior. Thus one should take into account the values of f around the line considered. Therefore the definition of directional local L^p regularity should include such information. However, in the asymptotic of small scales, the values taken into account should be localized more and more sharply around this line. These considerations motivated the following definition of Jaffard [24].

Definition 2. Let $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an m-uple of nonnegative real numbers. The average regularity $\widetilde{\alpha}$ is the harmonic mean of the α_n, i.e.,

$$\frac{1}{\widetilde{\alpha}} = \frac{1}{m}\sum_{n=1}^{m}\frac{1}{\alpha_n}. \tag{9}$$

The anisotropy indices are

$$\nu_n = \frac{\widetilde{\alpha}}{\alpha_n}. \tag{10}$$

Let $b = (e_1, \ldots, e_m)$ be an orthonormal basis of $\mathbb{R}^m$. Denote by $(x_1, \ldots, x_m)$ the coordinates of x on the basis b.

Set

$$\mathbf{v} = (v_1, \ldots, v_m) \quad \text{and} \quad |x|_{\mathbf{v}} = \max\left(|x_1|^{1/v_1}, \ldots, |x_m|^{1/v_m}\right). \tag{11}$$

Then $|\cdot|_{\mathbf{v}}$ is a quasi-norm on $\mathbb{R}^m$; i.e., it satisfies the requirements of a norm except for the triangular inequality which is replaced by the weaker condition

$$\exists C > 0 \ \forall x, y \in \mathbb{R}^m \quad \|x+y\|_{\mathbf{v}} \le C\left(\|x\|_{\mathbf{v}} + \|y\|_{\mathbf{v}}\right). \tag{12}$$

It is also continuous and homogeneous in the sense that

$$\forall r > 0 \ \forall x \in \mathbb{R}^m \quad \left|r^{v_1}x_1, \ldots, r^{v_m}x_m\right|_{\mathbf{v}} = r\,|x|_{\mathbf{v}}. \tag{13}$$

The corresponding $\mathbf{v}$-ball $B_{\mathbf{v},b}(x,r) := \{y \in \mathbb{R}^m : |x-y|_{\mathbf{v}} < r\}$ of $\mathbf{v}$-radius r centered on x is a rectangle with sides parallel to the axes of coordinates, centered at x and with side-length $2r^{v_n}$ in the x_n-direction.

Let $f : \mathbb{R}^m \longrightarrow \mathbb{C}$ be a function which belongs locally to L^p. One says that $f \in T^{p,\overrightarrow{\alpha}}(y,b)$ if there exist constants $C, 0 < R < 1$ and a polynomial $P(x) = \sum_{I=(i_1,\ldots,i_m)\in\mathbb{N}_0^m} a_I x^I = \sum_{I=(i_1,\ldots,i_m)\in\mathbb{N}_0^m} a_I x_1^{i_1} \ldots x_m^{i_m}$ of degree

$$d_J(P) < \overrightarrow{\alpha} \tag{14}$$

in the sense that

$$\max\left\{\sum_{n=1}^{m} \frac{i_n}{\alpha_n} : a_I \neq 0\right\} < 1 \tag{15}$$

such that

$$\forall r \le R \quad \|f(\cdot) - P(\cdot - y)\|_{L^p(B_{\mathbf{v},b}(y,r))} \le C r^{\widetilde{\alpha}+m/p}. \tag{16}$$

Remark 3. Actually, here $B_{\mathbf{v},b}(x,r)$ replaces the elongated ellipsoid $\mathcal{N}_{b,\overrightarrow{\alpha}}(x,r)$ of axis of lengths $2r^{v_1}, \ldots, 2r^{v_m}$, centered on x, considered in [24]. This ellipsoid corresponds to the ball centered at x with $\rho_{\mathbf{v}}$ radius r, where $\rho_{\mathbf{v}}$ is the quasi-norm on $\mathbb{R}^m$ defined by $\rho_{\mathbf{v}}(0) = 0$, and for $x \neq 0$, $\rho_{\mathbf{v}}(x)$ is the unique value of r for which $\sum_{n=1}^{m}(x_n/r^{v_n})^2 = 1$. Both $|\cdot|_{\mathbf{v}}$ and $\rho_{\mathbf{v}}$ are equivalent. Nevertheless, $|\cdot|_{\mathbf{v}}$ has the advantage of being easier in the computations of distances.

In the isotropic case ($v_n = 1$ for all $1 \le n \le m$), $\rho_{\mathbf{v}}$ coincides with the Euclidean norm.

We first extend the definition of $T^{p,\overrightarrow{\alpha}}(y,b)$ for an m-uple $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_m)$ of nonpositive real numbers as follows.

Definition 4. Let $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an m-uple of nonpositive real numbers. We say that $f \in T^{p,\overrightarrow{\alpha}}(y,b)$ if there exist constants C and $0 < R < 1$ such that

$$\forall r \le R \quad \|f\|_{L^p(B_{\mathbf{v},b}(y,r))} \le C r^{\widetilde{\alpha}+m/p}. \tag{17}$$

Contrary to the case $p = \infty$ studied in [24], there is no partial ordering property for $1 \le p < \infty$; i.e., if either $0 < \alpha_1 \le \beta_1, \ldots, 0 < \alpha_m \le \beta_m$ or $\alpha_1 \le \beta_1 < 0, \ldots, \alpha_m \le \beta_m < 0$ then $f \in T^{p,\overrightarrow{\beta}}(y,b)$ does not imply that $f \in T^{p,\overrightarrow{\alpha}}(y,b)$. Indeed, the harmonic mean values $\widetilde{\alpha}$ of $\overrightarrow{\alpha}$ and $\widetilde{\beta}$ of $\overrightarrow{\beta}$ satisfy $\widetilde{\alpha} \le \widetilde{\beta}$, but the corresponding anisotropy indices $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{u} = (u_1, \ldots, u_m)$ are not necessarily the same. Nevertheless, we will show that the following substitute for directional local L^p regularity $\alpha^{p,e}(y)$ given in (8) makes sense (see next section).

Definition 5. Let $e \in \mathbb{R}^m$ be a unit vector and b be any orthonormal basis starting with the vector e. Let $f : \mathbb{R}^m \longrightarrow \mathbb{C}$ be a function. Set

$$E^+(p,y,e) := \left\{\alpha_1 > 0 : \exists 0 < \varepsilon \le \alpha_1 \ f \in T^{p,\overrightarrow{(\alpha_1,\varepsilon,\ldots,\varepsilon)}}(y,b)\right\} \tag{18}$$

and

$$E^-(p,y,e) := \left\{\alpha_1 < 0 : \exists 0 < \varepsilon \le -\alpha_1 \ f \in T^{p,\overrightarrow{(\alpha_1,-\varepsilon,\ldots,-\varepsilon)}}(y,b)\right\}. \tag{19}$$

If $E^+(p,y,e) \neq \emptyset$, define the p–exponent of f at y in direction e by

$$\alpha(p,y,e) = \sup E^+(p,y,e). \tag{20}$$

If $E^+(p,y,e) = \emptyset$, define the p–exponent of f at y in direction e by

$$\alpha(p,y,e) = \sup E^-(p,y,e). \tag{21}$$

Remark 6. Clearly we can choose any orthonormal basis b starting with the vector e, in fact the component $x_1 - y_1$ is the same in any b and is equal to the inner product of $x - y$ with e, and all norms of $\mathbb{R}^{m-1}$ are equivalent.

Definition 7. Let $e \in \mathbb{R}^m$ be a unit vector. Let $f : \mathbb{R}^m \longrightarrow \mathbb{C}$ be a function. The directional L^p spectrum (resp., upper L^p spectrum) of f in direction e is the map which associates with each H the Hausdorff dimension $d_p(H,e)$ (resp., $D_p(H,e)$) of the set of points y where $\alpha(p,y,e) = H$ (resp., $\alpha(p,y,e) \le H$).

The paper is organized as follows. In Section 2, we first show that Definition 5 makes sense. Then we prove a criterion of directional pointwise p–exponent in terms of a supremum on a wide range of oriented anisotropic L^p regularities. We therefore deduce a first upper bound for the directional L^p spectra. In Section 3, we establish criteria of directional pointwise p–exponent in terms of decay condition on Triebel anisotropic wavelet coefficients and wavelet leaders. In Section 4, we obtain an alternative upper bound for the directional L^p spectra expressed as an infimum of Legendre transforms of some p–anisotropic scaling functions. Finally, in Section 5, the obtained results are illustrated by some examples of self-affine cascade functions.

2. Directional L^p Regularity Criterion

2.1. Approval of Definition 5. We first show that Definition 5 makes sense.

(i) Suppose that $f \in T^{p,\overrightarrow{(\beta_1,\varepsilon,\ldots,\varepsilon)}}(y,b)$ with $0<\varepsilon<\beta_1$. Let $\alpha_1<\beta_1$. Set $\varepsilon'=(\alpha_1/\beta_1)\varepsilon$. Then $\varepsilon'<\varepsilon$ and $\varepsilon'<\alpha_1$. Set $\overrightarrow{\alpha}:=\overrightarrow{(\alpha_1,\varepsilon',\ldots,\varepsilon')}$ and $\overrightarrow{\beta}:=\overrightarrow{(\beta_1,\varepsilon,\ldots,\varepsilon)}$. We will prove that $f\in T^{p,\overrightarrow{\alpha}}(y,b)$; clearly, the harmonic mean values $\widetilde{\alpha}$ of $\overrightarrow{\alpha}$ and $\widetilde{\beta}$ of $\overrightarrow{\beta}$ satisfy $\widetilde{\alpha}\le\widetilde{\beta}$. The corresponding anisotropy indices $\mathbf{v}=(v_1,\ldots,v_m)$ are the same.

Since $f\in T^{p,\overrightarrow{\beta}}(y,b)$ then there exist constants $C, 0<R<1$ and a polynomial $P(x)=\sum_{I=(i_1,\ldots,i_m)\in\mathbb{N}_0^m}a_I x^I$ of degree $d_J(P)$ less than $\overrightarrow{\beta}$ (see (14) and (15)), i.e.,

$$\max\left\{\frac{i_1}{\beta_1}+\frac{i_2}{\varepsilon}+\cdots+\frac{i_m}{\varepsilon}:a_I\neq 0\right\}<1, \tag{22}$$

such that

$$\begin{gathered}\forall r\le R\\ \|f(\cdot)-P(\cdot-y)\|_{L^p(B_{\mathbf{v},b}(y,r))}\le Cr^{\widetilde{\beta}+m/p}.\end{gathered} \tag{23}$$

Split P as the sum of two polynomials Q and S, where the indices I of the nonvanishing coefficients a_I of Q satisfy $i_1/\alpha_1+i_2/\varepsilon'+\cdots+i_m/\varepsilon'<1$. Therefore the indices I of the nonvanishing coefficients a_I of S satisfy both $i_1/\beta_1+i_2/\varepsilon+\cdots+i_m/\varepsilon<1$ and $i_1/\alpha_1+i_2/\varepsilon'+\cdots+i_m/\varepsilon'\ge 1$. Clearly $d_J(Q)<\overrightarrow{\alpha}$ and if $R<1$ then $r^{\widetilde{\beta}+m/p}\le Cr^{\widetilde{\alpha}+m/p}$. For each index I of the nonvanishing coefficients a_I of S

$$\begin{aligned}\int_{B_{\mathbf{v}}(y,r)}\left|(y-x)^I\right|^p dx&=\int_{B_{\mathbf{v}}(0,r)}\left|x^I\right|^p dx\\ &=\prod_{n=1}^{m}\int_{-r^{v_n}}^{r^{v_n}}\left|x_n\right|^{i_n p}dx_n\\ &\le C\prod_{n=1}^{m}r^{v_n(i_n p+1)}\le Cr^{\widetilde{\alpha}p+m}.\end{aligned} \tag{24}$$

It follows that $f\in T^{p,\overrightarrow{\alpha}}(y,b)$.

(ii) Suppose that $f\in T^{p,\overrightarrow{(\beta_1,-\varepsilon,\ldots,-\varepsilon)}}(y,b)$ with $0<\varepsilon<-\beta_1$. Let $\alpha_1<\beta_1$. Set $\varepsilon'=(\alpha_1/\beta_1)\varepsilon$. Then $-\varepsilon'<-\varepsilon$ and $0<\varepsilon'<-\alpha_1$. Clearly the harmonic mean values of $\overrightarrow{\alpha}:=\overrightarrow{(\alpha_1,-\varepsilon',\ldots,-\varepsilon')}$ and $\overrightarrow{\beta}:=\overrightarrow{(\beta_1,-\varepsilon,\ldots,-\varepsilon)}$ satisfy $\widetilde{\alpha}\le\widetilde{\beta}<0$. The corresponding anisotropy indices $\mathbf{v}=(v_1,\ldots,v_m)$ are the same. Since $f\in T^{p,\overrightarrow{\beta}}(y,b)$ then there exist constants C and $0<R<1$ such that

$$\begin{gathered}\forall r\le R\\ \|f(\cdot)\|_{L^p(B_{\mathbf{v}}(y,r))}\le Cr^{\widetilde{\beta}+m/p}\le Cr^{\widetilde{\alpha}+m/p}.\end{gathered} \tag{25}$$

It follows that $f\in T^{p,\overrightarrow{\alpha}}(y,b)$.

2.2. Criterion of Directional Pointwise p–Exponent. We will establish a criterion of directional pointwise p–exponent in terms of a supremum on a wide range of oriented anisotropic L^p regularities. The latest regularity is reminiscent of [40, 41] where Calderón and Torchinsky on one side and Folland and Stein on the other side have developed a theory of anisotropic $\mathscr{H}^p_{\mathbf{u}}(\mathbb{R}^m)$ spaces using the $\rho_{\mathbf{u}}$ quasi-norm of Remark 3.

Let $\mathbf{u}=(u_1,\ldots,u_m)\in\mathbb{R}^m$ be such that

$$\begin{gathered}\forall n\ u_n>0\\ \text{and }\sum_{n=1}^{m}u_n=m.\end{gathered} \tag{26}$$

We will say that $\mathbf{u}$ is an anisotropy vector.

If $I=(i_1,\ldots,i_m)\in\mathbb{N}_0^m$, set $d_{\mathbf{u}}(I)=\sum_{n=1}^m u_n i_n$. If $P=\sum_I a_I x^I$, $a_I\in\mathbb{C}$ is a polynomial, define its $\mathbf{u}$-homogeneous degree by

$$d_{\mathbf{u}}(P)=\max\left\{d_{\mathbf{u}}(I):a_I\neq 0\right\}. \tag{27}$$

Definition 8. Let $f:\mathbb{R}^m\longrightarrow\mathbb{C}$ be a function which belongs locally to L^p. Let b be an orthonormal basis of $\mathbb{R}^m$.

If $\alpha\ge 0$, we say that f belongs to $T^p_{\mathbf{u},\alpha}(y,b)$ if there exist constants $C, 0<R<1$ and a polynomial P of $\mathbf{u}$-homogeneous degree less than α such that

$$\begin{gathered}\forall r\le R\\ \|f(\cdot)-P(\cdot-y)\|_{L^p(B_{\mathbf{u},b}(y,r))}\le Cr^{\alpha+m/p}.\end{gathered} \tag{28}$$

If $-m/p\le\alpha<0$, we say that f belongs to $T^p_{\mathbf{u},\alpha}(y,b)$ if there exist constants C and $0<R<1$ such that

$$\begin{gathered}\forall r\le R\\ \|f\|_{L^p(B_{\mathbf{u},b}(y,r))}\le Cr^{\alpha+m/p}.\end{gathered} \tag{29}$$

Definitions 2 and 4 are related to the previous anisotropic regularity in L^p setting. Let $\overrightarrow{\alpha}=(\alpha_1,\ldots,\alpha_m)$ be an m-uple of either nonpositive or nonnegative real numbers. Let $\widetilde{\alpha}$ (resp., $\mathbf{v}=(v_1,\ldots,v_m)$) be the corresponding harmonic mean value as in (9) (resp., anisotropy indices as in (10)).

If $\overrightarrow{\alpha}$ is an m-uple of nonpositive real numbers then it follows from (29) and Definition 4 that

$$T^{p,\overrightarrow{\alpha}}(y,b)=T^p_{\mathbf{v},\widetilde{\alpha}}(y,b). \tag{30}$$

If $\overrightarrow{\alpha}$ is an m-uple of nonnegative real numbers then

$$\begin{gathered}d_{\mathbf{v}}(I)=\sum_{n=1}^{m}v_n i_n<\widetilde{\alpha}\Longleftrightarrow\\ \sum_{n=1}^{m}\frac{i_n}{\alpha_n}<1.\end{gathered} \tag{31}$$

It follows from (14) and (15) that

$$\begin{gathered}d_{\mathbf{v}}(P)<\widetilde{\alpha}\Longleftrightarrow\\ d_J(P)<\overrightarrow{\alpha}.\end{gathered} \tag{32}$$

We deduce that

$$T^{p,\overrightarrow{\alpha}}(y,b) = T^{p}_{\mathbf{v},\widetilde{\alpha}}(y,b). \tag{33}$$

The following theorem extends the result obtained for the directional Hölder regularity in [38] to the L^p setting. It has the advantage to cover unbounded functions and negative values.

Theorem 9 (let $m \geq 2$). *Let $e \in \mathbb{R}^m$ be a unit vector and b be any orthonormal basis starting with the vector e. Let $\mathscr{E}$ be the set of all anisotropy vectors $\mathbf{u}$ with $u_2 = \cdots = u_m$.*

Let $f \in L^p_{loc}(\mathbb{R}^m)$. Then the p–exponent of f at y in direction e is given by

$$\alpha(p,y,e) = \sup_{\mathbf{u}\in\mathscr{E}} \left(\frac{\alpha_{p,\mathbf{u}}(y,b)}{u_1} \right), \tag{34}$$

where $\alpha_{p,\mathbf{u}}(y,b)$ is the so-called anisotropic $p-\mathbf{u}$ regularity of f at y oriented in basis b, defined as

$$\alpha_{p,\mathbf{u}}(y,b) = \sup\left\{\widetilde{\alpha};\ f \in T^{p}_{\mathbf{u},\widetilde{\alpha}}(y,b)\right\}. \tag{35}$$

Proof (let $\mathbf{u} \in \mathscr{E}$). Assume that $f \in T^{p}_{\mathbf{u},\widetilde{\alpha}}(y,b)$. Take $\alpha_1 = \widetilde{\alpha}/u_1$ and for all $2 \leq n \leq m$ take $\alpha_n = \widetilde{\alpha}/u_2$. Set $\overrightarrow{\alpha} = (\alpha_1,\ldots,\alpha_m)$. Clearly, the anisotropy indices $\mathbf{v} = (v_1,\ldots,v_m)$ of $\overrightarrow{\alpha}$ coincide with $\mathbf{u}$.

(i) If $\widetilde{\alpha} < 0$ then

$$f \in T^{p}_{\mathbf{u},\widetilde{\alpha}}(y,b) \Longleftrightarrow \quad f \in T^{p,\overrightarrow{\alpha}}(y,b). \tag{36}$$

Since $\overrightarrow{\alpha} = (\alpha_1,\ldots,\alpha_m)$ with $0 > \alpha_m = \cdots = \alpha_2 \geq \alpha_1$, then it follows from Definition 5 that

$$\alpha(p,y,e) \geq \alpha_1 = \frac{\widetilde{\alpha}}{u_1}. \tag{37}$$

Then

$$\alpha(p,y,e) \geq \frac{1}{u_1} \sup\left\{\widetilde{\alpha};\ f \in T^{p}_{\mathbf{u},\widetilde{\alpha}}(y,b)\right\}. \tag{38}$$

Taking at the right the supremum over $\mathscr{E}$ yields the lower bound in (34).

(ii) If $\widetilde{\alpha} > 0$ then

$$d_{\mathbf{u}}(I) = \sum_{n=1}^{m} u_n i_n < \widetilde{\alpha} \Longleftrightarrow \quad \sum_{n=1}^{m} \frac{i_n}{\alpha_n} < 1. \tag{39}$$

So

$$d_{\mathbf{u}}(P) < \widetilde{\alpha} \Longleftrightarrow \quad d_J(P) < \overrightarrow{\alpha}. \tag{40}$$

We deduce that (36) holds too.

Since $\overrightarrow{\alpha} = (\alpha_1,\ldots,\alpha_m)$ with $0 < \alpha_2 = \cdots = \alpha_m \leq \alpha_1$, then it follows from Definition 5 that

$$\alpha(p,y,e) \geq \alpha_1 = \frac{\widetilde{\alpha}}{u_1}. \tag{41}$$

Then

$$\alpha(p,y,e) \geq \frac{1}{u_1} \sup\left\{\widetilde{\alpha};\ f \in T^{p}_{\mathbf{u},\widetilde{\alpha}}(y,b)\right\}. \tag{42}$$

Taking at the right the supremum over $\mathscr{E}$ yields the lower bound in (34).

Let us now prove the upper bound.

(i) Assume that $f \in T^{p,\overrightarrow{\alpha}}(y,b)$ with $\overrightarrow{\alpha} = (\alpha_1,\ldots,\alpha_m)$, $\alpha_2 = \cdots = \alpha_m = \varepsilon$ and $0 < \varepsilon \leq \alpha_1$. It follows from (30) that

$$\widetilde{\alpha} \leq \alpha_{p,\mathbf{v}}(y,b). \tag{43}$$

Then

$$\alpha_1 = \frac{\widetilde{\alpha}}{v_1} \leq \sup_{\mathbf{u}\in\mathscr{E}} \left(\frac{\alpha_{p,\mathbf{u}}(y,b)}{u_1} \right). \tag{44}$$

Thus

$$\alpha(p,y,e) \leq \sup_{\mathbf{u}\in\mathscr{E}} \left(\frac{\alpha_{p,\mathbf{u}}(y,b)}{u_1} \right). \tag{45}$$

Whence Theorem 9 holds.

(ii) The case where $f \in T^{p,\overrightarrow{\alpha}}(y,b)$ with $\overrightarrow{\alpha} = (\alpha_1,\ldots,\alpha_m)$ with $\alpha_2 = \cdots = \alpha_m = -\varepsilon$ and $0 < \varepsilon \leq -\alpha_1$ is similar. □

2.3. General Upper Bound for the Directional L^p Spectra. Let $\mathscr{E}$ be the set of anisotropy vectors $\mathbf{u}$ given in Theorem 9. Let b be an orthonormal basis of $\mathbb{R}^m$. Define the p-$\mathbf{u}$-sets of f in basis b by

$$E_{p,\mathbf{u}}(h,b) = \left\{x;\ \alpha_{p,\mathbf{u}}(x,b) = h\right\} \tag{46}$$

and the upper $p-\mathbf{u}$-sets of f in basis b by

$$E^{h,b}_{p,\mathbf{u}} = \left\{x;\ \alpha_{p,\mathbf{u}}(x,b) \leq h\right\}. \tag{47}$$

Theorem 9 yields the following general upper bound for the directional upper L^p spectrum given in Definition 7.

Corollary 10. *Let $\mathscr{E}$ be as in Theorem 9. Let $e \in \mathbb{R}^m$ be a unit vector and b be any orthonormal basis starting with the vector e. If $f \in L^p_{loc}(\mathbb{R}^m)$ then the directional L^p spectra of f in direction e given in Definition 7 satisfy*

$$\forall H \quad d_p(H,e) \leq D_p(H,e) \leq \inf_{\mathbf{u}\in\mathscr{E}} \dim E^{(u_1 H,b)}_{p,\mathbf{u}}, \tag{48}$$

where dim *denotes the Hausdorff dimension.*

3. Criteria of Directional L^p Regularity in Anisotropic Triebel Wavelet Bases

For each $\beta \in \mathbb{N}$ there are Daubechies [42] real valued compactly supported father and mother wavelets ψ_F and ψ_M in $C^\beta(\mathbb{R})$ (in the sense that they have classical continuous derivatives up to order β) such that $\int_{\mathbb{R}} \psi_F(x)dx = 1$, the moments $\int_{\mathbb{R}} x^n \psi_M(x)dx = 0$ for $n = 0,\ldots,\beta$, and the collection $(\psi_F(\cdot - k))_{k\in\mathbb{Z}}$ and $(2^{j/2}\psi_M(2^j \cdot -k))_{j\in\mathbb{N}_0,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Let $\mathbf{u}$ be an anisotropic vector. For $r > 0$, define the $\mathbf{u}$-anisotropy map as

$$r^{\mathbf{u}}x = (r^{u_1}x_1,\ldots,r^{u_m}x_m). \tag{49}$$

In [35, 36], Triebel has considered anisotropic multiresolution analysis; consider, for any $j \in \mathbb{N}_0$, the closed subspace $V_{j,\mathbf{u}}$ of $L^2(\mathbb{R}^m)$ spanned by the orthonormal basis $(2^{\sum_{n=1}^m [ju_n]/2}\Phi_{j,k,\mathbf{u}})_{k\in\mathbb{Z}^m}$, where

$$\Phi_{j,k,\mathbf{u}}(x) = \prod_{n=1}^m \psi_F\left(2^{[ju_n]}x_n - k_n\right). \tag{50}$$

The sequence $(V_{j,\mathbf{u}})_{j\in\mathbb{N}_0}$ is an anisotropic multiresolution analysis of $L^2(\mathbb{R}^m)$ in the sense that

(i) $\forall j \in \mathbb{N}_0,\ V_{j,\mathbf{u}} \subset V_{j+1,\mathbf{u}}$.

(ii) $f(x) \in V_{j,\mathbf{u}} \Longleftrightarrow f(2^{\mathbf{u}}x) \in V_{j+1,\mathbf{u}}$.

(iii) $\overline{\bigcup_{j\in\mathbb{N}_0} V_{j,\mathbf{u}}} = L^2(\mathbb{R}^m)$.

For $j \in \mathbb{Z}$, let $I_{j,\mathbf{u}}$ be the set of pairs $(G,\mathbf{l})$ where $G = (G_1,\ldots,G_m) \in \{F,M\}^m$ such that at least one component G_n is M and $\mathbf{l} = (l_1,\ldots,l_m) \in \mathbb{N}_0^m$ where

$$l_n = [ju_n] \quad \text{if } G_n = F, \tag{51}$$

$$\begin{aligned} &[ju_n] \le l_n < [(j+1)u_n] \quad \text{if } G_n = M \\ &\text{and } [(j+1)u_n] > [ju_n], \end{aligned} \tag{52}$$

and

$$\begin{aligned} &l_n = [ju_n] \quad \text{if } G_n = M \\ &\text{and } [(j+1)u_n] = [ju_n]. \end{aligned} \tag{53}$$

Clearly the cardinality $\sharp I_{j,\mathbf{u}}$ of $I_{j,\mathbf{u}}$ is bounded independently of j, more precisely

$$1 \le \sharp I_{j,\mathbf{u}} \le (2^m - 1)\prod_{n=1}^m (2+u_n). \tag{54}$$

The following proposition is given in [35, 36] in the case where b is the canonical basis of $\mathbb{R}^m$. It remains valid in the case where b is any orthonormal basis of $\mathbb{R}^m$.

Proposition 11. *Let b be an orthonormal basis of $\mathbb{R}^m$. Let $(x_1,\ldots,x_m)$ be the coordinates of x in b. Set*

$$\begin{aligned} \Phi_{k,b}(x) &:= \prod_{i=1}^m \psi_F(x_i - k_i), \\ \Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},b}(x) &= \prod_{i=1}^m \psi_{G_i}\left(2^{l_i}x_i - k_i\right) \\ \text{and } |\mathbf{l}| &:= \sum_{i=1}^m l_i. \end{aligned} \tag{55}$$

The collection of the union of $(\Phi_{k,b})$ for $k \in \mathbb{Z}^m$ and $(2^{|\mathbf{l}|/2}\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},b})$ for $j \in \mathbb{N}_0$, $(G,\mathbf{l}) \in I_{j,\mathbf{u}}$, and $k \in \mathbb{Z}^m$ is then an orthonormal basis of $L^2(\mathbb{R}^m)$. Thus any function $f \in L^2(\mathbb{R}^m)$ can be written as

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}^m} C_{k,b}\Phi_{k,b}(x) \\ &+ \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^m}\sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} c^{(G,\mathbf{l})}_{j,k,\mathbf{u},b}\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},b}(x), \end{aligned} \tag{56}$$

with

$$C_{k,b} = \int_{\mathbb{R}^m} f(x)\Phi_{k,b}(x)\,dx \tag{57}$$

and

$$c^{(G,\mathbf{l})}_{j,k,\mathbf{u},b} = 2^{|\mathbf{l}|}\int_{\mathbb{R}^m} f(x)\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},b}(x)\,dx. \tag{58}$$

A straightforward extension is given by the following result.

Proposition 12. *The collection of $(2^{|\mathbf{l}|/2}\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathscr{B}})$ for $j \in \mathbb{Z}$, $(G,\mathbf{l}) \in I_{j,\mathbf{u}}$, and $k \in \mathbb{Z}^m$ is an orthonormal basis of $L^2(\mathbb{R}^m)$. Thus any function $f \in L^2(\mathbb{R}^m)$ can be written as*

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^m}\sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} c^{(G,\mathbf{l})}_{j,k,\mathbf{u},b}\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},b}(x). \tag{59}$$

3.1. Wavelet Characterization of the $T^p_{\mathbf{u},\alpha}(y)$ Spaces.

Let $\mathbf{u}$ be an anisotropic vector. One of the fundamental properties of the anisotropic wavelet bases is that it characterizes $T^p_{\mathbf{u},\alpha}(y,b)$ spaces. The results are reminiscent of [5] where the authors have considered the isotropic case $u_1 = \cdots = u_m = 1$. *Without any loss of generality, we will present the results in the canonical basis and drop the letter b.* The proofs adapt all steps and arguments of [5] to the anisotropic setting, using

(i) the properties of the homogeneous norm $|\cdot|_{\mathbf{u}}$,

(ii) anisotropic versions of mean value theorem and Taylor's theorem with remainder (see [40, 41]),

(iii) anisotropic Triebel wavelet characterization of anisotropic Besov spaces $B^{s,q}_{p,\mathbf{u}}(\mathbb{R}^m)$ (see [35, 36]).

Proposition 13 (the **u**-mean value theorem). *Let* $\mathbf{u}$ *be an anisotropic vector. There exist two positive constants* C *and* ν *such that for all functions* f *of class* C^1 *on* $\mathbb{R}^m$ *and all* $x, y \in \mathbb{R}^m$,

$$\left|f(y) - f(x)\right| \leq C \sum_{i=1}^{m} \left|y - x\right|_{\mathbf{u}}^{u_i} \sup_{|h|_{\mathbf{u}} \leq \nu |y-x|_{\mathbf{u}}} \left|\partial_{x_i} f(x+h)\right|. \quad (60)$$

We denote by $\Delta_{\mathbf{u}}$ *the additive subsemigroup of* $\mathbb{R}$ *generated by* $0, u_1, u_2, \ldots$ *and* u_m. *In other words,* $\Delta_{\mathbf{u}}$ *is the set of all numbers* $d_{\mathbf{u}}(I)$ *as* I *ranges over* $\mathbb{N}_0^m$.

Proposition 14 (the **u**-Taylor inequality). *Let* $\mathbf{u}$ *be an anisotropic vector. Put* $u_{min} = \min_{1\leq n\leq m} u_n$. *Suppose* $\delta_{\mathbf{u}} \in \Delta_{\mathbf{u}}$, $\delta_{\mathbf{u}} > 0$, *and* $k = [\delta_{\mathbf{u}}/u_{min}]$. *There are two constants* $C_\delta > 0$ *and* $\nu > 0$ *such that for all functions* f *of class* $C^{(k+1)}$ *on* $\mathbb{R}^m$ *and all* $x, y \in \mathbb{R}^m$,

$$\left|f(y) - P(y-x)\right| \leq C_\delta \sum_{|I|\leq k+1, d_{\mathbf{u}}(I) > \delta_{\mathbf{u}}} \left|y - x\right|_{\mathbf{u}}^{d_{\mathbf{u}}(I)} \cdot \sup_{|h|_{\mathbf{u}} \leq \nu^{k+1} |y-x|_{\mathbf{u}}} \left|\partial^I f(x+h)\right| \quad (61)$$

where P *is the so-called* **u***-Taylor polynomial of* f *at* x *of* **u***-homogeneous degree* $\delta_{\mathbf{u}}$, *given by*

$$P(y-x) = \sum_{I: d_{\mathbf{u}}(I) \leq \delta_{\mathbf{u}}} \frac{\partial^I f(x)}{I!} (y-x)^I. \quad (62)$$

Proposition 15. *Let* $\mathbf{u}$ *be an anisotropic vector. Let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$. *Then* $f \in B^{s,q}_{p,\mathbf{u}}(\mathbb{R}^m)$ *if and only if its norm*

$$\|f\|_{B^{s,q}_{p,\mathbf{u}}(\mathbb{R}^m)} := \left(\sum_{k\in\mathbb{Z}^m} \left|C_k\right|^p\right)^{1/p} + \left(\sum_{j\in\mathbb{N}_0} \left(\sum_{k\in\mathbb{Z}^m} \sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} \left|2^{(s-m/p)j} c^{(G,\mathbf{l})}_{j,k,\mathbf{u}}\right|^p\right)^{q/p}\right)^{1/q} \quad (63)$$

is finite, with the usual modification if $p = \infty$ *and/or* $q = \infty$. *This characterization does not depend on the chosen (smooth enough) wavelets* ψ_F *and* ψ_M.

In [35] (section 5.3), a transference method has been proposed with the aim of showing that isotropic and anisotropic Besov spaces are isomorphic together and with the space $b^s_{p,q}$ of all union of sequences $(C_k)_{k\in\mathbb{Z}^m}$ and $(c_{j,k})_{j\in\mathbb{N}_0, k\in\mathbb{Z}^m}$ such that

$$\left(\sum_{k\in\mathbb{Z}^m} \left|C_k\right|^p\right)^{1/p} + \left(\sum_{j\in\mathbb{N}_0} \left(\sum_{k\in\mathbb{Z}^m} \left|2^{(s-m/p)j} c_{j,k}\right|^p\right)^{q/p}\right)^{1/q} < \infty, \quad (64)$$

with the usual modification if $p = \infty$ and/or $q = \infty$. In particular, any embedding theorem for isotropic Besov spaces can be carried over to anisotropic Besov spaces.

Definition 16. Let $\mathbf{u}$ be an anisotropic vector. Let s, s' be real number and let p and q be positive real numbers. One says that a function f belongs to $\dot{X}^{s,s',p,q}_{\mathbf{u}}(y)$ if there exists $A > 0$ such that

$$\varepsilon_j := 2^{j(s-m/p)} \left(\sum_{|y-2^{-\mathbf{l}}k|_{\mathbf{u}} \leq A} \sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} \left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u}}\right|^p \cdot \left(1 + 2^j \left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}\right)^{s'p}\right)^{1/p} \in \ell^q, \quad (65)$$

where

$$2^{-\mathbf{l}}k = \left(\frac{k_1}{2^{l_1}}, \ldots, \frac{k_m}{2^{l_m}}\right). \quad (66)$$

If $u_1 = \cdots = u_m = 1$ then this space was already introduced by Meyer and Xu [43] in order to study local oscillating behaviors.

Proposition 17. *Let* $\mathbf{u}$ *be an anisotropic vector. Let* $1 \leq p < \infty$, $0 \leq s \leq \beta$ *where* β *is the regularity of the Daubechies wavelets. Let* $y \in \mathbb{R}^m$ *and* $f \in L^p_{loc}(\mathbb{R}^m)$.

(i) *If* $f \in \dot{X}^{s,-s,p,1}_{\mathbf{u}}(y)$, *then* $f \in T^p_{\mathbf{u},s-m/p}(y)$.

(ii) *If* $f \in T^p_{\mathbf{u},s-m/p}(y)$, *then there exist* A *and* $C > 0$ *such that*

$$\forall j \quad 2^{j(sp-m)} \sum_{|y-2^{-\mathbf{l}}k|_{\mathbf{u}} \leq A} \sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} \left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u}}\right|^p \cdot \left(1 + 2^j \left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}\right)^{-sp} \leq Cj. \quad (67)$$

Let $p \geq 1$ and f be a function which belongs locally to L^p. For $A > 0$ small enough let

$$\Sigma^p_j(\mathbf{u}, s, A, y) = 2^{j(sp-m)} \left(\sum_{|y-2^{-\mathbf{l}}k|_{\mathbf{u}} \leq A} \sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} \left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u}}\right|^p \cdot \left(1 + 2^j \left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}\right)^{-sp}\right), \quad (68)$$

and

$$i_{\mathbf{u},p}(y) := \sup\left\{s : \liminf_{j\longmapsto\infty} \frac{\log\left(\Sigma^p_j(\mathbf{u}, s, A, y)^{1/p}\right)}{-j\log 2} \geq 0\right\}. \quad (69)$$

Proposition 18. *Let* $\mathbf{u}$ *be an anisotropic vector. Let* $p \geq 1$ *and* f *be a function which belongs locally to* $L^p(\mathbb{R}^m)$. *Then*

(1) $i_{\mathbf{u},p}(y)$ *is positive and independent of A,*

(2) *we have always*

$$\alpha_{p,\mathbf{u}}(y) \le i_{\mathbf{u},p}(y) - \frac{m}{p}, \tag{70}$$

(3) *if* $f \in B^{\delta,p}_{p,\mathbf{u}}(\mathbb{R}^m)$ *for some* $\delta > 0$ *then*

$$\alpha_{p,\mathbf{u}}(y) = i_{\mathbf{u},p}(y) - \frac{m}{p}. \tag{71}$$

Both Theorem 9 and Proposition 18 yield the following characterization of the pointwise directional p–exponent.

Corollary 19 (let $m \ge 2$). *Let* $e \in \mathbb{R}^m$ *be a unit vector and b be any orthonormal basis starting with the vector e. Let* $\mathscr{E}$ *be the set of all anisotropy vectors* $\mathbf{u}$ *such that* $u_2 = \cdots = u_m$.

Let $f \in L^p_{loc}(\mathbb{R}^m)$. *If* $f \in B^{\delta,p}_{p,\mathbf{u}}(\mathbb{R}^m)$ *for some* $\delta > 0$ *then the* p–*exponent of* f *at* y *in direction e is given by*

$$\alpha(p, y, e) = \sup_{\mathbf{u}\in\mathscr{E}} \left(\frac{i_{\mathbf{u},p}(y) - m/p}{u_1} \right). \tag{72}$$

3.2. Wavelet Leaders Characterization. Without any loss of generality, we will present the results in the canonical basis. Let $\mathbf{u}$ be an anisotropic vector. By $\lambda_{\mathbf{u}} = \lambda^{\mathbf{l}}_{j,k,\mathbf{u}}$ we denote a $\mathbf{u}$-dyadic rectangle in $\mathbb{R}^m$ of scale j, which has the form

$$\lambda_{\mathbf{u}} = \lambda^{\mathbf{l}}_{j,k,\mathbf{u}} = 2^{-\mathbf{l}}k + \prod_{i=1}^{m} \left[0, 2^{-l_i}\right), \tag{73}$$

where $2^{-\mathbf{l}}k$ was defined in (66).

Denote by $\Lambda_{j,\mathbf{u}}$ all $\mathbf{u}$-dyadic rectangles in $\mathbb{R}^m$ of scale j.

For $\lambda_{\mathbf{u}} \in \Lambda_{j,\mathbf{u}}$ set

$$\left|c_{\lambda_{\mathbf{u}}}\right| = \max \left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u}}\right|, \tag{74}$$

where the maximum is taken over all indices G giving the same $\mathbf{l}$ at scale j.

Let f be a function on $\mathbb{R}^m$. Define its $\mathbf{u}$ wavelet scaling function as

$$\eta_{\mathbf{u}}(p) = \liminf_{j\longrightarrow\infty} \frac{\log\left(2^{-mj}\sum_{\lambda\in\Lambda_{j,\mathbf{u}}}\left|c_{\lambda_{\mathbf{u}}}\right|^p\right)}{\log\left(2^{-j}\right)}. \tag{75}$$

By Proposition 15, this scaling function does not depend on the chosen (smooth enough) wavelets, and

$$\begin{gathered} \forall 0 < q \le \infty \\ \eta_{\mathbf{u}}(p) = \sup\left\{\tau : f \in B^{\tau/p,q}_{p,\mathbf{u}}(\mathbb{R}^m)\right\}. \end{gathered} \tag{76}$$

For $p > 0$ such that $\eta_{\mathbf{u}}(p) > 0$, the $p - \mathbf{u}$ wavelet leader of f at a rectangle $\lambda_{\mathbf{u}} \in \Lambda_{j,\mathbf{u}}$ is defined as

$$\ell^{(p)}_{\lambda_{\mathbf{u}}} = \left(\sum_{j'=j}^{\infty} \sum_{\lambda'_{\mathbf{u}}\in\Lambda_{j',\mathbf{u}},\lambda'_{\mathbf{u}}\subset 3\lambda_{\mathbf{u}}} \left|c_{\lambda'_{\mathbf{u}}}\right|^p 2^{-m(j'-j)} \right)^{1/p} \tag{77}$$

where $3\lambda_{\mathbf{u}}$ is the set formed by the rectangle $\lambda_{\mathbf{u}}$ and all its adjacent rectangles at scale j.

Remark 20. Note that if $\eta_{\mathbf{u}}(p) > 0$ then $f \in L^p_{loc}$ and if $\eta_{\mathbf{u}}(p) < 0$ then $f \notin L^p_{loc}$ (see [44]).

If $x \in \mathbb{R}^m$, denote by $\lambda_{j,\mathbf{u}}(x)$ the unique rectangle at the scale j that contains x.

As in [45], the third point in Proposition 18 yields the following result.

Proposition 21. *Let* $\mathbf{u}$ *be an anisotropic vector. Let* $p \ge 1$. *If* $\eta_{\mathbf{u}}(p) > 0$ *then*

$$\alpha_{p,\mathbf{u}}(x) = \liminf_{j\longrightarrow\infty} \frac{\log\left(\ell^{(p)}_{\lambda_{j,\mathbf{u}}(x)}\right)}{\log\left(2^{-j}\right)}. \tag{78}$$

Both Theorem 9 and Proposition 21 yield the following characterization of the pointwise directional p–exponent.

Corollary 22 (let $m \ge 2$). *Let* $e \in \mathbb{R}^m$ *be a unit vector and b any orthonormal basis starting with the vector e. Let* $\mathscr{E}$ *be the set of all anisotropy vectors* $\mathbf{u}$ *such that* $u_2 = \cdots = u_m$.

Let $p \ge 1$. *If* $\eta_{\mathbf{u}}(p) > 0$ *then the* p–*exponent of* f *at* y *in direction e is given by*

$$\alpha(p, y, e) = \sup_{\mathbf{u}\in\mathscr{E}} \left(\liminf_{j\longrightarrow\infty} \frac{\log\left(\ell^{(p)}_{\lambda_{j,\mathbf{u}}(x)}\right)}{\log\left(2^{-ju_1}\right)} \right). \tag{79}$$

4. Alternative Upper Bound of the Directional L^p Spectra

In Corollary 10, if $\mathbf{u}$ is an anisotropic vector different from $(1,\ldots,1)$, then the sets $E^{(u_1H,b)}_{p,\mathbf{u}}$ are anisotropic but dim is isotropic. Computing $\dim E^{(u_1H,b)}_{p,\mathbf{u}}$ is a very difficult task (see the example of Sierpinski carpet in [46] pages 118–119). To overcome this problem, in [47, 48], we adapted the notion of Hausdorff dimension to the anisotropy $\mathbf{u}$; if $\Omega \subset \mathbb{R}^d$, define its $\mathbf{u}$-diameter to be $|\Omega|_{\mathbf{u}} \coloneqq \sup_{x,y\in\Omega}|x-y|_{\mathbf{u}}$. By replacing in the definition of Hausdorff measure, the usual notion of diameter by the $\mathbf{u}$-diameter, we easily check (see [49]) that we get the following notion of anisotropic dimension.

Definition 23. Let $\Omega \subset \mathbb{R}^d$, $\varepsilon > 0$ and let R_ε be the set of all coverings $R = (\Omega_n)_{n\in\mathbb{N}}$ of Ω by sets Ω_n of $\mathbf{u}$-diameter $|\Omega_n|_{\mathbf{u}}$ at most ε. Let

$$M^\delta_{\varepsilon,\mathbf{u}}(\Omega) = \inf_{R\in R_\varepsilon} \sum_{n\in\mathbb{N}} \left|\Omega_n\right|^\delta_{\mathbf{u}}. \tag{80}$$

The δ-dimensional $\mathbf{u}$-Hausdorff measure of Ω is

$$M^\delta_{\mathbf{u}}(\Omega) = \limsup_{\varepsilon\longrightarrow 0} M^\delta_{\varepsilon,\mathbf{u}}(\Omega). \tag{81}$$

The $\mathbf{u}$-Hausdorff dimension of Ω is

$$\begin{aligned} \dim_{\mathbf{u}}(\Omega) &= \inf\left\{\delta : M^\delta_{\mathbf{u}}(\Omega) = 0\right\} \\ &= \sup\left\{\delta : M^\delta_{\mathbf{u}}(\Omega) = \infty\right\}. \end{aligned} \tag{82}$$

Note that we get the same value of $\dim_{\mathbf{u}}(\Omega)$ if we use coverings $R = (\Omega_n)_{n\in\mathbb{N}}$ of Ω by rectangles Ω_n with sides parallel to the axes of coordinates and with side-length $2\varepsilon^{\mathbf{u}_i}$ in the x_i-direction.

In the isotropic case, $|\cdot|_{(1,\ldots,1)}$ is equivalent to the Euclidean norm $|\cdot|$ on $\mathbb{R}^d$ and $\dim_{(1,\ldots,1)}(\Omega)$ coincides with $\dim\Omega$. But if $\mathbf{u} \neq (1,\ldots,1)$, then $\dim_{\mathbf{u}}(\Omega)$ does not necessarily coincide with $\dim\Omega$. Actually, if $u_{min} = \min_{1\le n\le m} u_n$ and $u_{max} = \max_{1\le n\le m} u_n$ there exists $C \ge 1$ such that

$$\forall x \in \mathbb{R}^d$$
$$\frac{1}{C}\min\left\{|x|^{1/u_{min}}, |x|^{1/u_{max}}\right\} \le |x|_{\mathbf{u}} \le C\max\left\{|x|^{1/u_{min}}, |x|^{1/u_{max}}\right\}. \tag{83}$$

and

$$u_{min}\dim(\Omega) \le \dim_{\mathbf{u}}(\Omega) \le u_{max}\dim(\Omega). \tag{84}$$

Definition 24. The $p-\mathbf{u}$-spectrum is defined as

$$d_{p,\mathbf{u}}(H) = \dim_{\mathbf{u}}\left\{x \in \mathbb{R}^m : \alpha_{p,\mathbf{u}}(x) = H\right\}. \tag{85}$$

The $p-\mathbf{u}$-upper-spectrum is defined as

$$D_{p,\mathbf{u}}(H) = \dim_{\mathbf{u}}\left\{x \in \mathbb{R}^m : \alpha_{p,\mathbf{u}}(x) \le H\right\}. \tag{86}$$

The restricted $p-\mathbf{u}$ wavelet leader of f at a rectangle $\lambda_{\mathbf{u}} \in \Lambda_{j,\mathbf{u}}$ is defined as

$$\widetilde{\ell}^{(p)}_{\lambda_{\mathbf{u}}} = \left(\sum_{j'=j}^{\infty}\sum_{\lambda'_{\mathbf{u}}\in\Lambda_{j',\mathbf{u}},\lambda'_{\mathbf{u}}\subset\lambda_{\mathbf{u}}}\left|c_{\lambda'_{\mathbf{u}}}\right|^p 2^{-m(j'-j)}\right)^{1/p}. \tag{87}$$

The $p-\mathbf{u}$ structure function is defined as

$$\widetilde{S}^{(p)}_{\mathbf{u}}(q,j) = 2^{-mj}\sum_{\lambda_{\mathbf{u}}\in\Lambda_{j,\mathbf{u}}}\left(\widetilde{\ell}^{(p)}_{\lambda_{\mathbf{u}}}\right)^q \tag{88}$$

The $p-\mathbf{u}$ wavelet leader scaling function is given by

$$\zeta^{(p)}_{\mathbf{u}}(q) = \liminf_{j\to\infty}\frac{\log\widetilde{S}^{(p)}_{\mathbf{u}}(q,j)}{\log(2^{-j})}. \tag{89}$$

We can easily adapt the (isotropic) general upper bound obtained in [45] or [50] to the anisotropic setting.

Proposition 25 (let $p \ge 1$). *If $\eta_{\mathbf{u}}(p) > 0$ then*

$$\forall h$$
$$D_{p,\mathbf{u}}(h) \le \inf_{q\in\mathbb{R}}\left(d + hq - \zeta^{(p)}_{\mathbf{u}}(q)\right). \tag{90}$$

Consequently, we obtain the following result.

Theorem 26 (let $m \ge 2$). *Let $e \in \mathbb{R}^m$ be a unit vector and b any orthonormal basis starting with the vector e. Let $\mathscr{E}$ be the set of all anisotropy vectors $\mathbf{u}$ such that $u_2 = \cdots = u_m$.*

Let $p \ge 1$. Then the directional L^p spectra of f in direction e given in Definition 7 satisfy

$$\forall H$$
$$d_p(H,e) \le D_p(H,e) \le \inf_{\mathbf{u}\in\mathscr{E},\eta_{\mathbf{u}}(p)>0}\frac{1}{u_{min}}\inf_{q\in\mathbb{R}}\left(\mathbf{u}_1 Hq - \zeta^{(p)}_{\mathbf{u}}(q) + m\right). \tag{91}$$

Proof. By (84)

$$D_p(H,e) \le \inf_{\mathbf{u}\in\mathscr{E}}\frac{1}{u_{min}}D_{p,\mathbf{u}}(\mathbf{u}_1 H). \tag{92}$$

Thus (90) yields (91). □

5. Examples of Affine Cascade Functions

We will apply Theorem 26 for some examples of self-affine cascade functions. Let s and t be two nonnegative integers. We divide the unit square $\mathfrak{R} = [0,1]^2$ into a uniform grid of rectangles of sides $1/t$ and $1/s$. Choose $A \subset \{0,1,\ldots,s-1\}\times\{0,1,\ldots,t-1\}$. For $\omega = (a,b) \in A$, the contraction $S_\omega(x_1,x_2) = (x_1/s + a/s, x_2/t + b/t)$ maps the unit square $\mathfrak{R}$ into the rectangle

$$\mathfrak{R}_\omega = \left[\frac{a}{s},\frac{a+1}{s}\right]\times\left[\frac{b}{t},\frac{b+1}{t}\right]. \tag{93}$$

If G is a subset of $\mathbb{R}^m$, we define the mapping S by

$$S(G) = \bigcup_{\omega\in A}S_\omega(G). \tag{94}$$

The Sierpinski carpet K (see [28, 29, 48]) and references therein) is the unique nonempty compact set satisfying

$$K = \bigcap_{n\in\mathbb{N}}S^n(\mathfrak{R}). \tag{95}$$

Let $(\gamma_\omega)_{\omega\in A}$ be nonnegative scalars. Put $g(x) = \psi_M(x_1)\psi_M(x_2)$ where ψ_M is a mother (smooth enough) wavelet. The Sierpinski cascade function adapted to the subdivision A satisfies the self-affine equation

$$\forall x \in \mathfrak{R}$$
$$F(x) = \sum_{\omega\in A}\gamma_\omega F\left(S_\omega^{-1}(x)\right) + g(x). \tag{96}$$

Proposition 27 (let $1 \le p < \infty$). *If $(\sum_{\omega\in A}\gamma_\omega)^p < st$, then the series*

$$F(x) = g(x) + \sum_{n=1}^{\infty}\sum_{(\omega_1,\ldots,\omega_n)\in A^n}\gamma_{\omega_1}\cdots\gamma_{\omega_n}g\left(S_{\omega_n}^{-1}\cdots S_{\omega_1}^{-1}(x)\right) \tag{97}$$

is the unique solution in $L^p(\mathfrak{R})$ for equation (96).

If moreover, $s = 2^S$ and $t = 2^T$ with S and T two nonnegative integers such that $S < T$ and $S+T$ is even and

(e_1, e_2) is the canonical basis in $\mathbb{R}^2$, then the directional L^p spectra of F in directions e_1 and e_2, respectively, satisfy

$$\forall H \quad d_p(H, e_1) \le D_p(H, e_1) \le \inf_{q \in \mathbb{R}} \left(Hq - \frac{\log \sum_{\omega \in A} \gamma_\omega^q}{\log s} \right) \tag{98}$$

and

$$\forall H \quad d_p(H, e_2) \le D_p(H, e_2) \le \inf_{q \in \mathbb{R}} \left(\frac{T}{S} Hq - \frac{\log \sum_{\omega \in A} \gamma_\omega^q}{\log s} \right). \tag{99}$$

Proof. Clearly series (97) satisfies (96). Since

$$\forall \omega \in A \quad \left\| F \circ S_\omega^{-1} \right\|_{L^p} = (st)^{1/p} \|F\|_{L^p}, \tag{100}$$

then assumption $(\sum_{\omega \in A} \gamma_\omega)^p < st$ implies that F is the unique solution in $L^p(\mathfrak{R})$.

Clearly if $\omega_l = (a_l, b_l)$ then

$$\begin{aligned} g\left(S_{\omega_n}^{-1} \cdots S_{\omega_1}^{-1}(x)\right) &= \psi_M\left(s^n x_1 - s^{n-1} a_1 - \cdots - s a_{n-1} - a_n\right) \\ &\cdot \psi_M\left(t^n x_2 - t^{n-1} b_1 - \cdots - t b_{n-1} - b_n\right). \end{aligned} \tag{101}$$

Let S and T be two positive integers such that $S < T$ and $S + T$ is even. Assume that $s = 2^S$ and $t = 2^T$. Put

$$u_1 = \frac{2S}{S+T} \quad \text{and } u_2 = \frac{2T}{S+T}. \tag{102}$$

Clearly $\mathbf{u} = (u_1, u_2)$ is an anisotropic vector with $u_1 < u_2$,

$$s^n = 2^{nS} = 2^{ju_1} \quad \text{and } t^n = 2^{nT} = 2^{ju_2} \tag{103}$$

with

$$j = \frac{n(S+T)}{2}. \tag{104}$$

By a straightforward computation, the $\mathbf{u}$ wavelet scaling function defined in (75) is given by

$$\eta_{\mathbf{u}}(p) = 2 - \frac{2 \log_2 \sum_{\omega \in A} \gamma_\omega^p}{S+T}. \tag{105}$$

Since $(\sum_{\omega \in A} \gamma_\omega)^p < st$, then $\sum_{\omega \in A} \gamma_\omega^p < st$. It follows that $\eta_{\mathbf{u}}(p) > 0$.

The restricted $p-\mathbf{u}$ wavelet leader of F at a rectangle $\lambda_{\mathbf{u}} \in \Lambda_{j,\mathbf{u}}$ (defined in (87)) satisfies

$$\left(\tilde{\ell}_{\lambda_{\mathbf{u}}}^{(p)}\right)^p = \sum_{n' \ge n} (st)^{(n-n')} \left(\sum_{\omega \in A} \gamma_\omega^p \right)^{n'-n} \left(\gamma_{\omega_1} \cdots \gamma_{\omega_n}\right)^p. \tag{106}$$

Since $\sum_{\omega \in A} \gamma_\omega^p < st$, then

$$\tilde{\ell}_{\lambda_{\mathbf{u}}}^{(p)} \sim \gamma_{\omega_1} \cdots \gamma_{\omega_n} \tag{107}$$

and

$$\zeta_{\mathbf{u}}^{(p)}(q) = 2 - \frac{2 \log_2 \sum_{\omega \in A} \gamma_\omega^q}{S+T}. \tag{108}$$

If (e_1, e_2) is the canonical basis in $\mathbb{R}^2$ then Theorem 26 yields

$$\forall H \quad d_p(H, e_1) \le D_p(H, e_1) \le \frac{1}{u_1} \inf_{q \in \mathbb{R}} \left(u_1 Hq - \zeta_{\mathbf{u}}^{(p)}(q) + 2 \right) \tag{109}$$

and

$$\forall H \quad d_p(H, e_2) \le D_p(H, e_2) \le \frac{1}{u_1} \inf_{q \in \mathbb{R}} \left(u_2 Hq - \zeta_{\mathbf{u}}^{(p)}(q) + 2 \right). \tag{110}$$

Using (102) and (108), we deduce results (98) and (99). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Mourad Ben Slimane, Borhen Halouani, and Farouq Alshormani extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group No. (RG-1435-063). Mourad Ben Slimane is thankful to Stéphane Jaffard for stimulating discussions.

References

[1] U. Frisch and G. Parisi, "Fully developped turbulence and intermittency," in *Proceedings of the International Summer School in Physics*, E. Fermi, Ed., pp. 84–88, Holland, 1985.

[2] A. Arneodo, E. Bacry, and J. F. Muzy, "Singularity spectrum of fractal signals from wavelet analysis: exact results," *Journal of Statistical Physics*, vol. 70, no. 3-4, pp. 635–674, 1993.

[3] A. Calderón and A. Zygmund, “Local properties of solutions of elliptic partial differential equations,” *Studia Mathematica*, vol. 20, no. 2, pp. 171–227, 1961.

[4] S. Jaffard, “Pointwise regularity criteria,” *Comptes Rendus Mathematique*, vol. 339, no. 11, pp. 757–762, 2004.

[5] S. Jaffard and C. Mélot, “Wavelet analysis of fractal boundaries. Part 1: local exponents,” *Communications in Mathematical Physics*, vol. 258, no. 3, pp. 513–539, 2005.

[6] M. Ben Slimane and C. Mélot, “Analysis of a fractal boundary: the graph of the knopp function,” *Abstract and Applied Analysis*, vol. 2015, Article ID 587347, 14 pages, 2015.

[7] Y. Heurteaux and S. Jaffard, “Multifractal analysis of images: new connexions between analysis and geometry,” in *Proceedings of the NATO-ASI Conference on Imaging for Detection and Identification*, Springer, Netherlands, 2006.

[8] C. Tricot, “General Hausdorff functions, and the notion of one-sided measure and dimension,” *Arkiv for Matematik*, vol. 48, no. 1, pp. 149–176, 2010.

[9] P. Abry, M. Clausel, S. Jaffard, S. G. Roux, and B. Vedel, “Hyperbolic wavelet transform: an efficient tool for multifractal analysis of anisotropic textures,” *Revista Matematica Iberoamericana*, vol. 31, no. 1, pp. 313–348, 2015.

[10] P. Abry, S. G. Roux, H. Wendt et al., “Multiscale anisotropic texture analysis and classification of photographic prints: art scholarship meets image processing algorithms,” *IEEE Signal Processing Magazine*, vol. 32, no. 4, pp. 18–27, 2015.

[11] H. Aimar and I. Gómez, “Parabolic Besov regularity for the heat equation,” *Constructive Approximation*, vol. 36, no. 1, pp. 145–159, 2012.

[12] A. Arneodo, B. Audit, N. Decoster, J.-F. Muzy, and C. Vaillant, “Wavelet-based multifractal formalism: applications to DNA sequences, satellite images of the cloud structure and stock market data,” in *The Science of Disasters*, A. Bunde, J. Kropp, and H. J. Schellnhuber, Eds., pp. 27–102, Springer, Berlin, Germany, 2002.

[13] M. Clausel and B. Vedel, “Explicit construction of operator scaling Gaussian random fields. Fractals,” *Complex Geometry, Patterns, and Scaling in Nature and Society*, vol. 19, no. 1, pp. 101–111, 2011.

[14] L. Ponson, D. Bonamy, H. Auradou et al., “Anisotropic self-affine properties of experimental fracture surfaces,” *International Journal of Fracture*, vol. 140, no. 1-4, pp. 27–37, 2006.

[15] S. G. Roux, M. Clausel, B. Vedel, S. Jaffard, and P. Abry, “Self-Similar anisotropic texture analysis: the hyperbolic wavelet transform contribution,” *IEEE Transactions on Image Processing*, vol. 22, no. 11, pp. 4353–4363, 2013.

[16] J. Sampo and S. Sumetkijakan, “Estimations of hölder regularities and direction of singularity by hart smith and curvelet transforms,” *Journal of Fourier Analysis and Applications*, vol. 15, no. 1, pp. 58–79, 2009.

[17] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, vol. 18, North-Holland, Amsterdam, The Netherlands, 1978.

[18] E. J. Candès and D. L. Donoho, “Ridgelets: a key to higher-dimensional intermittency?” *Philosophical Transactions of the Royal Society A: Mathematical, Physical & Engineering Sciences*, vol. 357, no. 1760, pp. 2495–2509, 1999.

[19] S. Mallat, “Applied mathematics meets signal processing,” in *Proceedings of the Challenges for the 21st century. Papers from the international conference on fundamental sciences: mathematics and theoretical physics (ICFS 2000)*, L. H. Y. Chen, Ed., pp. 138–161, World Scientific, Singapore, 2001.

[20] D. L. Donoho, “Wedgelets: nearly minimax estimation of edges,” *The Annals of Statistics*, vol. 27, no. 3, pp. 859–897, 1999.

[21] K. Guo and D. Labate, “Analysis and detection of surface discontinuities using the 3D continuous shearlet transform,” *Applied and Computational Harmonic Analysis* , vol. 30, no. 2, pp. 231–242, 2011.

[22] P. Lakhonchai, J. Sampo, and S. Sumetkijakan, “Shearlet transforms and Hölder regularities,” *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 8, no. 5, pp. 743–771, 2010.

[23] K. Nualtong and S. Sumetkijakan, “Analysis of Höder regularities by wavelet-like transforms with parabolic scaling,” *Thai Journal of Mathematics*, pp. 275–283, 2005.

[24] S. Jaffard, “Pointwise and directional regularity of nonharmonic Fourier series,” *Applied and Computational Harmonic Analysis* , vol. 28, no. 3, pp. 251–266, 2010.

[25] H. Biermé, M. M. Meerschaert, and H. Scheffler, “Operator scaling stable random fields,” *Stochastic Processes and Their Applications*, vol. 117, no. 3, pp. 312–332, 2007.

[26] A. Bonami and A. Estrade, “Anisotropic analysis of some Gaussian models,” *Journal of Fourier Analysis and Applications*, vol. 9, no. 3, pp. 215–236, 2003.

[27] S. Davies and P. Hall, “Fractal analysis of surface roughness by using spatial data,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 61, no. 1, pp. 3–37, 1999.

[28] J. King, “The Singularity Spectrum for General Sierpinski Carpets,” *Advances in Mathematics*, vol. 116, no. 1, pp. 1–11, 1995.

[29] L. Olsen, “Self-affine multifractal Sierpinski sponges in Rd. Pacific,” *Journal of Mathematics*, vol. 183, no. 1, pp. 143–199, 1998.

[30] W. Farkas, “Atomic and subatomic decompositions in anisotropic function spaces,” *Mathematische Nachrichten*, vol. 209, pp. 83–113, 2000.

[31] G. Garrigós and A. Tabacco, “Wavelet decompositions of anisotropic Besov spaces,” *Mathematische Nachrichten*, vol. 239, pp. 80–102, 2002.

[32] G. Garrigós, R. Hochmuth, and A. Tabacco, “Wavelet characterizations for anisotropic Besov spaces with $0 < p < 1$,” *Proceedings of the Edinburgh Mathematical Society*, vol. 47, no. 3, pp. 573–595, 2004.

[33] R. Hochmuth, “Wavelet characterizations for anisotropic besov spaces,” *Applied and Computational Harmonic Analysis*, vol. 12, no. 2, pp. 179–208, 2002.

[34] A. Kamont, “On the fractional anisotropic Wiener field,” *Probability and Mathematical Statistics*, vol. 16, no. 1, pp. 85–98, 1996.

[35] H. Triebel, *Theory of Function Spaces III, Monographs in Mathematics*, vol. 78, Birkhäuser, Basel, Switzerland, 2006.

[36] H. Triebel, “Wavelet bases in anisotropic function spaces,” in *Proceedings of the Function Spaces, Differential Operators And Nonlinear Analysis*, pp. 370–387, Prague, Czechia, 2004 (Arabic).

[37] M. Ben Slimane, “Anisotropic two-microlocal spaces and regularity,” *Journal of Function Spaces*, vol. 2014, Article ID 505796, 10 pages, 2014.

[38] M. Ben Slimane and H. Ben Braiek, “Directional and anisotropic regularity and irregularity criteria in Triebel wavelet bases,” *Journal of Fourier Analysis and Applications*, vol. 18, no. 5, pp. 893–914, 2012.

[39] M. Ben Slimane and H. Ben Braiek, “Baire generic results for the anisotropic multifractal formalism,” *Revista Matemática Complutense*, vol. 29, no. 1, pp. 127–167, 2016.

[40] A.-P. Calderón and A. Torchinsky, "Parabolic maximal functions associated with a distribution," *Advances in Mathematics*, vol. 24, no. 2, pp. 101–171, 1977.

[41] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, vol. 28 of *Mathematical Notes*, Princeton University Press, Princeton, NJ, USA, 1982.

[42] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 909–996, 1988.

[43] Y. Meyer and H. Xu, "Wavelet analysis and chirps," *Applied and Computational Harmonic Analysis*, vol. 4, no. 4, pp. 366–379, 1997.

[44] S. Jaffard, C. Melot, R. Leonarduzzi et al., "p-exponent and p-leaders, Part I: Negative pointwise regularity," *Physica A: Statistical Mechanics and its Applications*, vol. 448, pp. 300–318, 2016.

[45] S. Jaffard and C. Mélot, "Wavelet analysis of fractal boundaries. Part 2: multifractal analysis," *Communications in Mathematical Physics*, vol. 258, no. 3, pp. 541–565, 2005.

[46] K. Falconer, *Fractal Geometry: Mathematical Foundations And Applications*, John Wiley and sons, Toronto, Canada, 1990.

[47] H. Ben Braiek and M. Ben Slimane, "Directional regularity criteria," *Comptes Rendus Mathematique*, vol. 349, no. 7-8, pp. 385–389, 2011.

[48] M. Ben Slimane, "Multifractal formalism and anisotropic self-similar functions," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 124, no. 2, pp. 329–363, 1998.

[49] C. A. Rogers, *Hausdorff Measures*, Cambridge University Press, Cambridge, UK, 1970.

[50] R. Leonarduzzi, H. Wendt, P. Abry et al., "p-exponent and p-leaders, Part II: Multifractal analysis. Relations to detrended fluctuation analysis," *Physica A: Statistical Mechanics and its Applications*, vol. 448, pp. 319–339, 2016.