

Question: 1 (a) Find polynomial function $p(x) = ax^2 + bx + c$ such that

$$p(0) = 1, p(1) = 4, p(2) = 11. \quad [8]$$

(b) Use the reduced row echelon form to solve the system of linear equations

$$x + y + z - 3w = -2$$

$$2x + 3y - 4z = 1$$

$$-3x - 4y - z + 6w = -1$$

[8]

Question: 2 (a)

If $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, find the inverse of A by using Elementary matrix method. [8]

(b) If A is a 3x3 matrix and $\det A = 2$,

find $\det(4A)$, $\det(4A^{-1})$ and $\det(A^4)$

[6]

Question: 3 Solve the system of linear equations by Cramer's Rule

$$x + y + z = 7$$

$$-x + y + z = 5$$

$$x - y + z = 5$$

[8]

Question: 4 (a) Let $a = \langle 2, 1, 0 \rangle$, $b = \langle 2, -1, 3 \rangle$

(i) Find the angle between a and b

[6]

(ii) Compute the projection of b onto a

(b) If $u = \langle 1, 1, 1 \rangle$, $v = \langle 3, -1, 1 \rangle$, find value of $\alpha \in \mathbb{R}$, if

$$\|\alpha u - v\| = \sqrt{20}$$

[6]

M - 107

KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
(SEMESTER II, 1433-1434) FIRST MID-TERM

FULL MARKS: 50

TIME: 90min

Question: 1 (a) Find polynomial function $p(x) = ax^2 + bx + c$ such that $p(0) = 1, p(1) = 4, p(2) = 11$.

[8]

Solution: Given Polynomial function is

$$p(x) = ax^2 + bx + c \quad \text{Eq.1}$$

at $x = 0$, we obtained $p(0) = 0 + 0 + c = 1$

at $x = 1$, we obtained $p(1) = a + b + c = 4$

at $x = 2$, we obtained $p(2) = 4a + 2b + c = 11$

We obtained system of linear equations

$$c = 1$$

$$a + b + c = 4$$

$$4a + 2b + c = 11$$

Writing the system in the matrix form

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 11 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 11 \end{bmatrix}$$

Solving by using Gauss Jordan method, Augmented matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 11 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 11 \\ 0 & 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence $a = 2, b = 1, c = 1$ Substituting these values in Eq.1.

The required polynomial function is $p(x) = 2x^2 + x + 1$

(b) Use the reduced row echelon form to solve the system of linear equations

$$x + y + z - 3w = -2$$

$$2x + 3y - 4z = 1$$

$$-3x - 4y - z + 6w = -1$$

[8]

Solution: writing the system of equation in matrix form ,

$$\begin{bmatrix} 1 & 1 & 1 & -3 \\ 2 & 3 & -4 & 0 \\ -3 & -4 & -1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

Solving by using Reduced row echelon form Row echelon form ,Augmented matrix is

$$\approx \begin{bmatrix} 1 & 1 & 1 & -3 & -2 \\ 2 & 3 & -4 & 0 & 1 \\ -3 & -4 & -1 & 6 & -1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & -3 & -2 \\ 0 & 1 & -6 & 6 & 5 \\ 0 & -1 & 2 & -3 & -7 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & -3 & -2 \\ 0 & 1 & -6 & 6 & 5 \\ 0 & 0 & -4 & 3 & -2 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & -3 & -2 \\ 0 & 1 & -6 & 6 & 5 \\ 0 & 0 & 1 & \frac{-3}{4} & \frac{1}{2} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 7 & -9 & -7 \\ 0 & 1 & -6 & 6 & 5 \\ 0 & 0 & 1 & \frac{-3}{4} & \frac{1}{2} \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & \frac{-15}{4} & \frac{-21}{2} \\ 0 & 1 & 0 & \frac{3}{2} & 8 \\ 0 & 0 & 1 & \frac{-3}{4} & \frac{1}{2} \end{bmatrix}$$

Writing the augmented matrix is the system of equation form

$$x - \frac{15}{4}w = \frac{-21}{2}$$

$$y - \frac{3}{2}w = 8$$

$$z - \frac{3}{4}w = \frac{1}{2}$$

The system will have infinite number of solutions

x , y , and z are leading variables and w is free variable, let $w = t$, where t is any real number

$$x = \frac{-21}{2} + \frac{15}{4}w$$

$$y = 8 - \frac{3}{2}w$$

$$z = \frac{1}{2} + \frac{3}{4}w$$

solutions are

$$x = \frac{-21}{2} + \frac{15}{4}t$$

$$y = 8 - \frac{3}{2}t$$

$$z = \frac{1}{2} + \frac{3}{4}t$$

$$w = t, \text{ where } t \in \mathbb{R}$$

Question: 2 (a)

If $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, find the inverse of A by using Elementary matrix method. [8]

Solution: Using Elementary matrix method to find A^{-1}

$$[A: I] = \left[\begin{array}{ccc|ccc} 2 & 1 & -3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \approx \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right] R_1 \leftrightarrow R_2$$

$$-2R_1 + R_2,$$

$$-R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -7 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \approx \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 7 & -1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right],$$

$$-2R_2 + R_3,$$

$$-\frac{1}{13}R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 7 & -1 & 2 & 0 \\ 0 & 0 & -13 & 2 & -4 & 1 \end{array} \right] \approx \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 7 & -1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{4}{13} & \frac{-1}{13} \end{array} \right]$$

$$\begin{array}{c} -7R_3 + R_2, -2R_3 + R_1 \\ -R_2 + R_1 \end{array} \approx \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{4}{13} & \frac{5}{13} & \frac{2}{13} \\ 0 & 1 & 0 & \frac{1}{13} & \frac{-2}{13} & \frac{7}{13} \\ 0 & 0 & 1 & \frac{-2}{13} & \frac{4}{13} & \frac{-1}{13} \end{array} \right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{13} & \frac{4}{13} & \frac{-5}{13} \\ 0 & 1 & 0 & \frac{1}{13} & \frac{-2}{13} & \frac{7}{13} \\ 0 & 0 & 1 & \frac{-2}{13} & \frac{4}{13} & \frac{-1}{13} \end{array} \right] = [I : A^{-1}]$$

$$\text{Hence } A^{-1} = \begin{bmatrix} \frac{3}{13} & \frac{4}{13} & \frac{-5}{13} \\ \frac{1}{13} & \frac{-2}{13} & \frac{7}{13} \\ \frac{-2}{13} & \frac{4}{13} & \frac{-1}{13} \end{bmatrix}$$

(b) If A is a 3×3 matrix and $\det A = 2$,
find $\det(4A)$, $\det(4A^{-1})$ and $\det(A^4)$

[6]

Solution: Using properties of determinant

$$(i) \det(4A) = 4^3 \det A = 64(2) = 128$$

$$(ii) \det(4A^{-1}) = 4^3 \det A^{-1} = \frac{64}{\det A} = \frac{64}{2} = 32$$

$$(iii) \det(A^4) = (\det A)^4 = (2)^4 = 16$$

Question: 3 Solve the system of linear equations by Cramer's Rule

$$x + y + z = 7$$

$$-x + y + z = 5$$

$$x - y + z = 5$$

[8]

Solution: Matrix form of the system of linear equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$$

$$\det A = \det \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = 4, \det A_1 = \det \begin{bmatrix} 7 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & -1 & 1 \end{bmatrix} = 4, \det A_2 = \det \begin{bmatrix} 1 & 7 & 1 \\ -1 & 5 & 1 \\ 1 & 5 & 1 \end{bmatrix} = 4$$

$$\det A_3 = \det \begin{bmatrix} 1 & 1 & 7 \\ -1 & 1 & 5 \\ 1 & -1 & 5 \end{bmatrix} = 20$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{4} = 1$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{4}{4} = 1$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{20}{4} = 5$$