M - 107

(SEMESTER II, 1430-1431) FIRST MID-TERM

Question: 1 (a) Solve the system of linear equations by Gaussian - elimination method **[8]**

$$x_1 + 2x_2 - x_3 = 1$$
$$x_1 + 3x_2 - 2x_3 = 2$$
$$-x_1 + x_2 + 4x_3 = 2.$$

Solution: Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 3 & -2 & 2 \\ -1 & 1 & 4 & 2 \end{bmatrix} \xrightarrow{-R_1+R_2,\, R_1+R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Backward substitution

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_2 - x_3 &= 1 \\ 2x_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -2x_2 + x_3 + 1 \\ x_2 &= x_3 + 1 \\ x_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_3 &= 0 \\ x_2 &= 0+1 = 1 \\ x_1 &= 1-2(1) = -1 \end{aligned}$$

Solution of the system is $x_1 = -1,\ x_2 = 1,\ x_3 = 0$

(b) Let A be an invertible matrix such that inverse of 4A is $\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$. **[6]**

Solution:

$$(4A)^{-1} = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} \text{ OR } \frac{1}{4}A^{-1} = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} \text{ OR } A^{-1} = 4\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 8 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 16 & 0 \\ 8 & 4 \end{bmatrix} \qquad A = \frac{1}{64}\begin{bmatrix} 4 & 0 \\ -8 & 16 \end{bmatrix}$$

$$f(A) = \frac{1}{(64)^2}\begin{bmatrix} 4 & 0 \\ -8 & 16 \end{bmatrix}^2 - \frac{2}{64}\begin{bmatrix} 4 & 0 \\ -8 & 16 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{(64)^2}\begin{bmatrix} 16 & 0 \\ -160 & 256 \end{bmatrix} - \frac{1}{32}\begin{bmatrix} 4 & 0 \\ -8 & 16 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{256} & 0 \\ -\frac{10}{256} & \frac{1}{16} \end{bmatrix} - \begin{bmatrix} \frac{1}{8} & 0 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{256} - \frac{1}{8} + 3 & 0 \\ -\frac{10}{256} + \frac{1}{4} & \frac{1}{16} - \frac{1}{2} + 3 \end{bmatrix}$$

Question: 2 (a) Let

$$x_1 - x_2 = 4$$
$$2x_1 + 2x_2 + x_3 = -1$$
$$3x_1 + 2x_2 + x_3 = 3.$$

[10]

Write the above system of linear equations in the form *AX=B* ,

i. **Find A^{-1} using elementary row operations,**
ii. **Use A^{-1} to solve the above system of equations.**

Solution: (i)

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \Rightarrow AX = B$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

(ii) $\left[A|I\right] \approx$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & -2 & 1 & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{5}{4} & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 5 & -4 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & -4 \end{bmatrix} \approx \left[I|A^{-1}\right]$$

(iii) $X = A^{-1}B$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & -1 & 1 \\ 2 & 5 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -9 \end{bmatrix}$$

Question: 3 (a) Solve the system of linear equations by Cramer's Rule **[8]**

$$x \qquad - z = 1$$
$$2x + y + z = 0$$
$$-x + 2y + 3z = 2.$$

Solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow AX = B$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}, \det(A) = -4, A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \det(A_1) = 3$$

$$A_2 = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 2 & 3 \end{bmatrix}, \det(A_2) = -13, A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix}, \det(A_3) = 7$$

By using Crammer's Rule

$$x = \frac{\det(A_1)}{\det(A)} = \frac{3}{-4} = -\frac{3}{4}, y = \frac{\det(A_2)}{\det(A)} = \frac{-13}{-4} = \frac{13}{4}, z = \frac{\det(A_3)}{\det(A)} = \frac{7}{-4} = -\frac{7}{4}$$

(b) Evaluate the following determinant **[8]**

$$\begin{vmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 \\ -1 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & 4 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -8 & 0 \\ 1 & -3 & 0 \\ 3 & 4 & 4 \end{vmatrix} = 4(9+8) = 68$$