KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
(SEMESTER II, 1433-1434)

M - 107
FULL MARKS :80
FINAL
TIME: 3hours

Question: 1 . (a) Given

[6+ 5+5]

$$\begin{aligned} 5x+4y \quad &= 132 \\ 5x+y+2z &= 132 \\ x+11y+2z &= 132 \end{aligned}$$

i. **Write the system of linear equations in the matrix AX=b ,**
ii. **Use method of cofactors to find A^{-1} , where A is coefficient matrix, and**
iii. **Use A^{-1} to solve the given system.**

Solution:

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$$i.\begin{bmatrix} 5 & 4 & 0 \\ 5 & 1 & 2 \\ 1 & 11 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 132 \\ 132 \\ 132 \end{bmatrix}$$

$$ii.A = \begin{bmatrix} 5 & 4 & 0 \\ 5 & 1 & 2 \\ 1 & 11 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{132}\begin{bmatrix} 20 & 8 & -8 \\ 8 & -10 & 10 \\ -54 & 51 & 15 \end{bmatrix}$$

$$iii.x = \frac{1}{132}\begin{bmatrix} 20 & 8 & -8 \\ 8 & -10 & 10 \\ -54 & 51 & 15 \end{bmatrix}\begin{bmatrix} 132 \\ 132 \\ 132 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ 12 \end{bmatrix}$$

(b) Let A be a 2 X 2 matrix such that the inverse of 6A is $\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$.Find A.

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$$(6A)^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$$

$$\frac{1}{6}A^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$$

$$A^{-1} = 6\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

Taking inverse of bothe the sides

$$A = \frac{1}{4-6}\begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Use Cramer's rule to solve the system

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$$\begin{aligned} x+y+z &= 3 \\ x-y+2z &= 2 \\ y+z &= 2 \end{aligned}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -3, |A_1| = \begin{vmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = -3, |A_2| = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -3$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = -3, \quad \text{solution is } x = 1, \quad y = 1, \quad z = 1.$$

Question: 2.(a) **Show that the line** $x = 5-6t,\ y = -3+4t,\ z = 4+2t$ **and the plane** $3x+2y+5z = 7$
[5+ 6+5] **are parallel. Hence find the distance between the line and the plane.**

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➔ **Vector parallel to line is** $v_1 = < -6, 4, 2 >$

Vector normal to plane is $v_2 = <3,2,5>$

If the line and plane are parallel then $v_1.v_2 = 0$

$v_1.v_2 = -18+8+12 = 0.$

➔ **Let a point on the plane is A (0,1,1) and point on line is B (5,-3,4)**

➔ **Vector between plane and line is** $AB = < 5, 4, 3 >$

➔ **Component of AB along the normal to plane is**

➔ $$comp_{v_2} AB = \frac{AB.v_2}{\|v_2\|} = \frac{5-8+15}{\sqrt{9+4+25}} = \frac{22}{\sqrt{38}}$$

(b) Find the equation of the line of intersection of the planes,
$x+y-z = 2$ and $2x-y+3z = 1$ **through the point P(-1,2,1) .**

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Normal to plane P_1 **is** $N_1 = <1,1,-1>$ **Normal to plane** P_2 **is** $N_2 = <2,-1,3>$
Normal vector to both plane is $N_3 = N_1 \text{x } N_2 = <2,-5,3>$
Let the point P (0,1,1) lies on both the planes
Find the equation of the line of intersection of the planes through the point P(-1,2,1) with parallel vector $N_3 = <2,-5,3>$

$$x = -1+ 2t, \quad y = 2 - 5t, \quad z = 1 + 3t$$

(c) Find the equation of the tangent plane to the graph $z = 4x^2 + y$ **at the point (-1, 5, 9)**

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$F(x, y, z) = 4x^2 + y - z = 0$

$\nabla F(x, y, z) = < 8x, 1, -1 >$

$\nabla F(-1, 5, 9) = < -8, 1, -1 >$ is vector nornal to the plane.

Equation of the plane is

$-8(x+1) + 1.(y-5) - 1.(z-9) = 0$

$-8x + y - z - 4 = 0$

Question: 3. (a) **A particle is moving along the curve** $x = t^2, \quad y = \frac{1}{2}t^2 - 4t, \quad z = 2t + 4$. **Find the component of velocity and acceleration at** $t = 2$ **in the direction of** $b = i + 2j - 3k$.

[6+ 6+4]

Solution:

6

$$r(t) = < t^2, \tfrac{1}{2}t^2 - 4t, 2t + 4 >$$

$$\text{Velocity:} \quad v(t) = \frac{dr}{dt} = < 2t, t - 4, 2 >$$

$$\text{Accelration}: a(t) = \frac{d^2r}{dt^2} = < 2, 1, 0 >$$

$$\text{at} \quad t = 2, \quad v(2) = < 4, -2, 2 >$$

$$a(2) = < 2, 1, 0 >$$

$$b = < 1, 2, -3 >$$

$$comp_b^v = \frac{v.b}{\|b\|} = \frac{4 - 4 - 6}{\sqrt{1 + 4 + 9}} = \frac{-6}{\sqrt{14}}$$

$$comp_b^a = \frac{a.b}{\|b\|} = \frac{2 + 2}{\sqrt{1 + 4 + 9}} = \frac{4}{\sqrt{14}}$$

(b)Find the unit tangent vector and principal normal vector for the curve C $r(t) = e^t \cos t \; i + e^t \sin t \; j + e^t k$

Solution:

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$$r(t) = e^t \cos ti + e^t \sin tj + e^t k$$

$$r'(t) = (e^t \cos t - e^t \sin t)i + (e^t \cos t + e^t \sin t)j + e^t k$$

$$\|r'(t)\| = \sqrt{3}e^t$$

$$\text{Unit tanget vector } T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{(e^t \cos t - e^t \sin t)i + (e^t \cos t + e^t \sin t)j + e^t k}{\sqrt{3}e^t}$$

$$= \frac{1}{\sqrt{3}}[(\cos t - \sin t)i + (\cos t + \sin t)j + k]$$

$$T'(t) = \frac{1}{\sqrt{3}}[(-\sin t - \cos t)i + (\sin t - \cos t)j]$$

$$\|T'(t)\| = \sqrt{\frac{2}{3}}$$

$$\text{Pricipal Normal vector } N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{1}{\sqrt{2}}[(-\sin t - \cos t)i + (\sin t - \cos t)j]$$

(c) Find the curvature of parabola $y = \frac{1}{4}x^2$ at the points *x = 0 and x =1.*

Solution:

$$y = \tfrac{1}{4}x^2, \quad y' = \tfrac{1}{2}x, \quad y'' = \tfrac{1}{2}$$

$$\text{Curvature:} \quad \kappa = \frac{\|y''\|}{\left[1+(y')^2\right]^{\frac{3}{2}}}$$

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$$\kappa = \frac{\frac{1}{2}}{\left[1+\left(\frac{1}{2}x\right)^2\right]^{\frac{3}{2}}}$$

$$at \qquad t = 0, \qquad \kappa = \frac{1}{2}$$

$$at \qquad t = 1, \qquad \kappa = \frac{\frac{1}{2}}{[1.25]^{\frac{3}{2}}} \approx 0.358$$

Question: 4. (a) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos y}{3x^2+y^2}$ **does not exist along the coordinate axes.**

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2+y^2}$$

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1. Along x-axis, $y=0$. $\lim\limits_{(x,0)\to(0,0)} \dfrac{x0\cos 0}{3x^2+0^2} = \lim\limits_{(x,0)\to(0,0)} \dfrac{0}{3x^2} = 0,$ 2

2. Along y-axis, $x=0$. $\lim\limits_{(0,y)\to(0,0)} \dfrac{0y\cos y}{30^2+y^2} = \lim\limits_{(0,y)\to(0,0)} \dfrac{0}{y^2} = 0$ ~~2~~

3. Along $y=x$. $\lim\limits_{(x,x)\to(0,0)} \dfrac{xx\cos x}{3x^2+x^2} = \lim\limits_{(x,x)\to(0,0)} \dfrac{x^2\cos x}{4x^2} = \lim\limits_{(x,x)\to(0,0)} \dfrac{\cos x}{4} = \dfrac{1}{4}$ 2

From 1 and 3 $\Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos y}{3x^2+y^2}$ does not exist 1

[5+ 5+8] **(b) Show that the function** $u(x,y) = x^2 - y^2 + 1$

satisfies the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Solution:

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$$u(x,y) = x^2 - y^2 + 1$$

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y \quad 2$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = -2 \quad 2$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 1$$

(c) Use differential to find approximate ... $\dfrac{xy}{x^2+y^2+1}$,

if the point $(1,2)$ change to $(1.03, 1.99)$

Solution:

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$$f(x,y) = \frac{xy}{x^2+y^2+1}$$

$$dx = 1.03 - 1.00 = 0.03, \quad dy = 1.99 - 2.00 = -0.01$$

$$f_x = \frac{y^3 - y - x^2 y}{\left(x^2+y^2+1\right)^2} = \frac{8+2-2}{(1+4+1)^2} = \frac{8}{36} \quad 2$$

$$f_y = \frac{x^3 - x - xy^2}{\left(x^2+y^2-1\right)^2} = \frac{1+1-4}{(1+4+1)^2} = \frac{-2}{36} \quad 2$$

2

$$df = f_x dx + f_y dy = \left(\frac{8}{36}\right)(0.03) + \left(\frac{-2}{36}\right)(-.01) \quad \cancel{2}\ 1$$

$$df = 0.072 \quad 2$$

Question: 5. (a) Find the relative extrema and saddle points, if any, of the function
[7+ 7] $f(x,y)=3x^2+6xy+7y^2-2x+4y$.

Solution:

$f(x,y)=3x^2+6xy+7y^2-2x+4y$

$f_x=6x+6y-2$

$f_y=6x+14y+4$ 1

To find Critical points

$f_x=6x+6y-2=0$

$f_y=6x+14y+4=0$ 2 3

Solving the above equation, we obtained $x=\frac{13}{12}, y=\frac{-3}{4}$

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Critical point is $\left(\frac{13}{12},\frac{-3}{4}\right)$

$f_{xx}=6, f_{yy}=14, f_{xy}=6$

$D(x,y)=84-36>0$, there is no saddle point,

$D\left(\frac{13}{12},\frac{-3}{4}\right)=84-36>0$ 2 4

$f_{xx}=6>0,$

hence, there is relative minima at point $\left(\frac{13}{12},\frac{-3}{4}\right)$ 2

(b) Use Lagrange multipliers to find the extreme value of the function
$f(x,y)=xy$ **subject to constraint** $g(x,y)=x^2+y^2-10=0$.

Solution:

$f(x,y)=xy,\qquad g(x,y)=x^2+y^2-10=0.$

1 $\nabla f(x,y)=<y,x>,\qquad \nabla g(x,y)=<2x,2y>$

Lagrange multiplier equation $\nabla f(x,y)=\lambda\nabla g(x,y)$

$<y,x>=\lambda<2x,2y>$

$x=2\lambda y$ .1

7 $y=2\lambda x$ .2 2 4

$x^2+y^2-10=0.$ .3

$x=2\lambda(2\lambda x)=4\lambda^2x$

$x-4\lambda^2x=0$

$x(1-4\lambda^2)=0\Rightarrow x=0$ or $1-4\lambda^2=0,$ $\lambda=\pm\frac{1}{2}$

$x\neq 0\ \Rightarrow\ 1-4\lambda^2=0\ \Rightarrow\ \lambda=\pm\frac{1}{\sqrt{2}}$ 1

From 1 and 2 $\frac{x}{y}=\frac{y}{x}\Rightarrow x^2=y^2$

From equation 3, $2x^2=10, x=\pm\sqrt{5}\Rightarrow y=\pm\sqrt{5}$

There are four points $(\sqrt{5},\sqrt{5})$, $(-\sqrt{5},\sqrt{5})$, $(\sqrt{5},-\sqrt{5})$, $(-\sqrt{5},-\sqrt{5})$ 3

$f(\sqrt{5},\sqrt{5})=5$

$f(-\sqrt{5},\sqrt{5})=-5$

$f(\sqrt{5},-\sqrt{5})=-5$ 2

$f(-\sqrt{5},-\sqrt{5})=5$

hence maximum value is 5, and mininum value is -5

NOTE: YOU ARE FREE TO CHANGE THE MARKING SCHEME FOR YOUR QUESTION, BUT JUST DISCUSS WITH ME