**TIME: 3 Hours** **KING SAUD UNIVERSITY** **FULL MARKS: 100**

**M - 107** **DEPARTMENT OF MATHEMATICS**

**FINAL EXAMINATION (SECOND SEMESTER 1432-1433)**

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**Question:** 1: (a) Find the value of y (only) by using Cramer's rule

$$x+2y-3z=1$$
$$0x+y+2z=2 \qquad [6]$$
$$-x+y-2z=-2$$

(b) Let $B=\begin{bmatrix}1&2&3\\2&1&3\\1&2&4\end{bmatrix}$ be a matrix. Find adj($B$) and hence find $B^{-1}$ if it exists. [10]

(c) Given points $O(0,0,0)$, $P(1,-1,2)$, $Q(0,3,-1)$ and $R(1,2,2)$. Find the volume of the box having adjacent sides $OP$, $OQ$, and $OR$. [6]

**Question:2** ( a) Determine whether the lines

$L_1: x=2t-1, y=t+2, z=-2t+4$ and $L_2: x=-s+3, y=s+1, z=s$

are parallel, skew, or intersect. If they intersect, find the point of intersection. [8]

(b) Consider the planes $2x+y+8z=8$ and $x+3y-z=-1$, find

(i) the angle between these planes, and

(ii) the parametric equations of the line of intersection of these planes. [10]

(c) Use Chain rule to find $\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}$ where $w=u^2+v^2$, $u=2x+y$ and $v=xy$. [8]

**Question: 3 (a)** Find the domain of the function $f(x,y)=\sqrt{\dfrac{x^2+y^2}{x^2-y^2}}$ and

show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\sqrt{\dfrac{x^2+y^2}{x^2-y^2}}$ does not exist. [8]

**(b)** Let $r(t)=\cos(4t)i+\sin(4t)j+3tk$ be the position vector of a particle at time t.

Find the tangential and the normal components of acceleration at time $t=\dfrac{\pi}{8}$. [10]

**(c)** Find equation of the tangent plane and the normal line to the paraboloid $z=24-3x^2-4y^2$ at the point $(2,1,8)$. [10]

**Question: 4 .(a)** Find the directional derivative of $f(x,y,z)=x^2+yz+z^2+1$ at the point (1,-1,2) in the direction of the normal to the plane $x-y+2z=6$. [10]

**(b)** Find local extrema and saddle points , if any, on the surface

$$x^2+2y^2-4x+4y+10=0. \qquad [8]$$

**(c)** Use method of Langrange multiplier to find extrema of the function $f(x,y,z)=x^2+y^2+z^2$ , subject to constraint $x-y+z=1$. [8]

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1) a)

$$y = \frac{\det A_2}{\det A}, \quad A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 2 \\ -1 & -2 & -2 \end{bmatrix}$$

$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (1)(-4) - (2)(2) + (-3)(1) = -11$

$\det A_2 = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (1)(0) - (1)(2) + (-3)(2) = -8$

$$y = \frac{\det A_2}{\det A} = \frac{-8}{-11} = \frac{8}{11}$$

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b)

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

$C_{11} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2, \quad C_{12} = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -5$

$C_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \quad C_{21} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = -2, \quad C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

$C_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0, \quad C_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3, \quad C_{32} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = +3$

$C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$

Matrix of Cofactors: $C = \begin{bmatrix} -2 & -5 & 3 \\ -2 & 1 & 0 \\ 3 & 3 & -3 \end{bmatrix}$

$$\operatorname{adj}(B) = C^t = \begin{bmatrix} -2 & -2 & 3 \\ -5 & 1 & 3 \\ 3 & 0 & -3 \end{bmatrix}$$

$\det B = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (1)(-2) + (2)(-5) + (3)(3) = -3 \neq 0$

$$B^{-1} = \frac{1}{\det B} \operatorname{adj}(B) = \frac{1}{-3} \begin{bmatrix} -2 & -2 & 3 \\ -5 & 1 & 3 \\ 3 & 0 & -3 \end{bmatrix}$$

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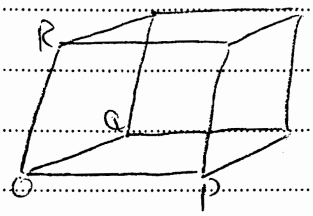
(c) Volume of box $= \|a \times b\| \|c\| \cos\theta = |(a \times b) \cdot c|$

$\overrightarrow{OP} = \langle 1, -1, 2 \rangle$

$\overrightarrow{OQ} = \langle 0, 3, -1 \rangle$

$\overrightarrow{OR} = \langle 1, 2, 2 \rangle$



Volume of box $= |(\overrightarrow{OP} \times \overrightarrow{OQ}) \cdot \overrightarrow{OR}|$

$\cancel{|(\overrightarrow{OP} \times \overrightarrow{OQ}) \cdot \overrightarrow{OR}| =}$

$$\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 0 & 3 & -1 \end{vmatrix} = -5i + j + 3k$$

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Volume $= |(\overrightarrow{OP} \times \overrightarrow{OQ}) \cdot \overrightarrow{OR}| = \langle -5, 1, 3 \rangle \cdot \langle 1, 2, 2 \rangle$

$= -5 + 2 + 6 = 3$ unit$^3$

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Q2 (a) $l_1: x = 2t - 1,\ y = t + 2,\ z = -2t + 4$

$l_2: x = -s + 3,\ y = s + 1,\ z = s$

we take the vectors which are parallel to lines:

$a = \langle 2, 1, -2 \rangle$

$b = \langle -1, 1, 1 \rangle$

$\frac{2}{-1} \neq \frac{1}{1}$ So the lines not parallel

See if they are intersected or not, for $P_0(x_0, y_0, z_0)$

$2t - 1 = -s + 3 \longrightarrow (1)$

$t + 2 = s + 1 \longrightarrow (2)$

$-2t + 4 = s \longrightarrow (3)$

$E_2 - E_3$

$t + 2 = s + 1$

$-2t + 4 = s$

$t + 2 + 2t - 4 = \cancel{s} + 1 - \cancel{s}$

$3t - 2 = 1 \Rightarrow 3t = 3$

$\boxed{t = 1}$

$s = -2t + 4 = -2 + 4 = 2$

$\boxed{s = 2}$

Check in Eq (1):

$2(1) - 1 = -(2) + 3$

$1 = 1$ ✓

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Substituting in $l_1$: $x = 2(t) - 1 = 2(1) - 1 = 1$

$y = t + 2 = 1 + 2 = 3$

$z = -2t + 4 = -2(1) + 4 = 2$

~~Point of~~ ∴ The lines are intersected and the

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(b) (i) $2x + y + 8z = 8 \; , \; x + 3y - z = -1$

$n_1 = \langle 2, 1, 8 \rangle \; , \; n_2 = \langle 1, 3, -1 \rangle$

$\|n_1\| = \sqrt{4+1+64} = \sqrt{69} \; , \|n_2\| = \sqrt{1+9+1} = \sqrt{11}$

$\theta = \cos^{-1} \frac{n_1 \cdot n_2}{\|n_1\|\|n_2\|} = \cos^{-1} \frac{\langle 2,1,8 \rangle \cdot \langle 1,3,-1 \rangle}{\sqrt{69}\;\sqrt{11}} = \cancel{\frac{\quad}{\sqrt{69}\sqrt{11}}}$

$\theta = \cos^{-1} \frac{2+3-8}{\sqrt{69}\;\sqrt{11}} = \cos^{-1} \frac{-3}{\sqrt{69}\sqrt{11}} \approx 96.3^\circ$

(ii) vector parallel to that line is: $n_1 \times n_2$

$$\vec{A} = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 8 \\ 1 & 3 & -1 \end{vmatrix} = -25i + 10j + 5k$$

we have the vector, but Also we ~~_~~ have to find point, So let $z = 0$, So:

$2x + y = 8 \rightarrow (1)$

$x + 3y = -1 \rightarrow (2)$

$-3E_1 + E_2$

$-6x - 3y = -24$

$x + 3y = -1$

$-5x = -25$

$x = 5$

substitute in $E_1$:

$2(5) + y = 8$

$\boxed{y = -2}$

So the point is $(5, -2, 0)$

Eq. of line passing through $p(5, -2, 0)$ and parallel to vector $\vec{A} = \langle -25, 10, 5 \rangle$ is:

$x = 5 - 25t$

$y = -2 + 10t$

$z = 5t \; , \; t \in \mathbb{R}$

T.O.P

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$\boxed{C}$ $w = u^2 + v^2$ , $u = 2x + y$ , $v = xy$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$= (2u)(2) + (2v)(y)$

Substitute by $u = 2x+y$ , $v = xy$ , :

$= (2(2x+y))(2) + (2(xy))(y)$

$= (4x+2y)(2) + (2xy)(y)$

$= 8x + 4y + 2xy^2$

8

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

$= (2u)(1) + (2v)(x)$

Substitute by $u = 2x+y$ , $v = xy$ , :

$= 4x + 2y + 2x^2y$

Q3 : $\boxed{a}$

Domain is:

$$\frac{x^2+y^2}{x^2-y^2} \geq 0$$

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Domain of the ~~...~~ domn.

$x^2 - y^2 \neq 0$

D : $x : R - \{x = y\}$ ; $y : R - \{y = x\}$

$$\lim_{(x,y)\to(0,0)} \sqrt{\frac{x^2+y^2}{x^2-y^2}}$$

there are ~~...~~ two ways of approaching $(0,0)$, ~~about~~ x-axis ($y=0$) at ~~x~~ y-axis ($x=0$)

$(y=0)$ $\lim_{(x,0)\to(0,0)} \sqrt{\frac{x^2+y^2}{x^2-y^2}} = \sqrt{\frac{x^2+0}{x^2-0}} = \sqrt{1}$

$(x=0)$ $\lim_{(0,y)\to(0,0)} \sqrt{\frac{x^2+y^2}{x^2-y^2}} = \sqrt{\frac{0+y^2}{0-y^2}} = \sqrt{-1} \rightarrow$ Does not Exist

So the limit DNE

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~~(b)~~ $\cancel{a_T = \frac{r'(t)\cdot r''(t)}{\|r'(t)\|}}$ , $\cancel{a_N = \frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|}}$

$\cancel{r(t) = (\cos 4t)\, i + (\sin 4t)\, j + 3t\, K}$

$\cancel{r'(t) = -4\sin 4t\, i + 4\cos 4t\, j + 3K}$

$\cancel{r''(t) = -16\cos 4t\, i - 16\sin 4t\, j + 0K}$

$\cancel{r'(t)\cdot r''(t) = \langle -4\sin 4t, 4\cos 4t, 3\rangle \cdot \langle -16\cos 4t, -16\sin 4t, 0\rangle}$

$\cancel{= 64\sin 4t\cos 4t - 64\sin 4t\cos 4t + 0 = 0}$

$$\cancel{r'(t)\times r''(t) = \begin{vmatrix} i & j & K \\ -4\sin 4t & 4\cos 4t & 3 \\ -16\cos 4t & -16\sin 4t & 0 \end{vmatrix}}$$

$\cancel{= (-48\sin 4t)\, i + 48\cos 4t\, j}$

(b) $a_T = \frac{r'(t)\cdot r''(t)}{\|r'(t)\|}$ , $a_N = \frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|}$

$\cancel{r(t) = \cos 4t\, i + \sin 4t\, j + 3t K}$

$\cancel{r(\frac{\pi}{8}) = 0 i + 1 j + \frac{3\pi}{8} K}$

$r(t) = \cos 4t\, i + \sin 4t\, j + 3t K$

$r'(t) = -4\sin 4t\, i + 4\cos 4t\, j + 3K$

$r''(t) = -16\cos 4t\, i - 16\sin 4t\, j$

at $t = \frac{\pi}{8}$ :

$r'(\frac{\pi}{8}) = -4 i + 0 j + 3K = \langle -4, 0, 3\rangle$

$r''(\frac{\pi}{8}) = 0 i - 16 j = \langle 0, -16, 0\rangle$

$r'(\frac{\pi}{8})\cdot r''(\frac{\pi}{8}) = 0 + 0 + 0 = 0$

$$r'(\tfrac{\pi}{8})\times r''(\tfrac{\pi}{8}) = \begin{vmatrix} i & j & K \\ -4 & 0 & 3 \\ 0 & -16 & 0 \end{vmatrix} = 48 i - 0 j + 64 K$$

$\|r'(\frac{\pi}{8})\times r''(\frac{\pi}{8})\| = \sqrt{48^2 + 64^2} = 80$

$\|r'(\frac{\pi}{8})\| = \sqrt{16 + 9} = 5$

$a_T = \frac{r'(\frac{\pi}{8})\cdot r''(\frac{\pi}{8})}{\|r'(\frac{\pi}{8})\|} = \frac{0}{5} = 0$

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$a_N = \frac{\|r'(\frac{\pi}{8})\times r''(\frac{\pi}{8})\|}{\|r'(\frac{\pi}{8})\|} = \frac{20}{5} = 16$

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b) $x^2 + 2y^2 - 4x + 4y + 10 = 0$

$f_x = 2x - 4$

$f_{xx} = 2$

$f_y = 4y + 4$

$f_{yy} = 4$

$f_{xy} = 0$

Discrement : $D = f_{xx} f_{yy} - (f_{xy})^2 = 8 > 0$

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$f_x = 2x - 4 = 0 \Longrightarrow 2x = 4 \Longrightarrow \boxed{x = 2}$

$f_y = 4y + 4 = 0 \Longrightarrow 4y = -4 \Longrightarrow \boxed{y = -1}$

there is only one critical point $(2, -1)$

at $(2, -1)$:

$D = 8 > 0$

$f_{xx} = 2 > 0$ , so $f(2,-1) = 4$ is local ~~maximum~~ minimum

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c) $f(x,y,z) = x^2 + y^2 + z^2$ function

$g(x,y,z) = x - y + z - 1 = 0$ Condition

$\nabla f_{(x,y,z)} = \lambda \nabla g_{(x,y,z)}$

$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, -1, 1 \rangle$

$2x = \lambda \rightarrow (1) \qquad x = \frac{\lambda}{2}$

$2y = -\lambda \rightarrow (2) \Longrightarrow \qquad y = \frac{-\lambda}{2}$ substitute in E4!

$2z = \lambda \rightarrow (3) \qquad z = \frac{\lambda}{2}$

$x - y + z - 1 = 0 \rightarrow (4)$

$(\frac{\lambda}{2}) - (\frac{-\lambda}{2}) + (\frac{\lambda}{2}) - 1 = 0$

$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = 1$

$\frac{3\lambda}{2} = 1$

$\boxed{\lambda = \frac{2}{3}}$ Substitute in $E_1, E_2, E_3$

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$x = \frac{\lambda}{2} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$

$y = \frac{-\lambda}{2} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3}$ $\Longrightarrow P(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$

$z = \frac{\lambda}{2} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$ only one critical point

$f(\frac{1}{3}, \frac{-1}{3}, \frac{1}{3}) = \frac{1}{3}$ max. value