TIME: 3 Hours **KING SAUD UNIVERSITY** **FULL MARKS: 100**

M - 107 **DEPARTMENT OF MATHEMATICS**

FINAL EXAMINATION (FIRST SEMESTER 1433-1434)

Question: 1: (a) For what values of λ the following system of equations

[6+6+8]

$$x+(\lambda-1)y=0$$
$$(\lambda-1)x+y=0$$

has (a) unique solution, and (b) infinitly many solutions.

(b) Let A be a 3x3 matrix, use properties of determinant to evaluate

$$\det(3A)+3\det(A).\det(A^{-1})+\det\left[(3A)^{-1}\right], \ \det(A)=-3.$$

(c) Let $B=\begin{bmatrix}1&2&1\\2&4&1\\3&7&0\end{bmatrix}$ be a matrix. Find adj(B) and use it to find B^{-1}, if exists.

Solution (a) Augmented Matrix is

$$\left[\begin{array}{cc|c}1 & \lambda-1 & 0\\ \lambda-1 & 1 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & \lambda-1 & 0\\ 0 & 1-(\lambda-1)^2 & 0\end{array}\right] \Rightarrow \left[1-(\lambda-1)^2\right]y=0$$

$y \neq 0$ because if $y=0$, then $x=0$ is trivial solution

$\Rightarrow 1-(\lambda-1)^2=0$ or $1-\lambda^2+2\lambda-1=0 \Rightarrow \lambda(2-\lambda)=0 \Rightarrow \lambda=0$ or $\lambda=2$

i) if $\lambda=2$, then $x+y=0 \Rightarrow x=-y$, there are infinitly many solutions

ii) if $\lambda=0$ then $x-y=0 \Rightarrow x=y$, there are infinity many solutions

iii) if $\lambda\neq0$, $\lambda\neq2$, then $x=0, y=0$ is unique solution.

(b) $\det(3A)+3\det(A)\cdot\det(A^{-1})+\det\left[(3A)^{-1}\right]$

$$=3^3\cdot(-3)+3\cdot(-3)\cdot\left(\frac{1}{-3}\right)+\frac{1}{3^3\cdot(-3)}$$

$$=-81+3-\frac{1}{81}=-\frac{6319}{81}$$

(c) Matrix of cofactors $C=\begin{bmatrix}-7&3&2\\7&-3&-1\\-2&1&0\end{bmatrix}$

$$\text{Adj}(B)=\begin{bmatrix}-7&7&2\\3&-3&1\\2&-1&0\end{bmatrix}. \qquad \det B=1.$$

$$B^{-1}=\frac{1}{\det B}\text{Adj } B=\begin{bmatrix}-7&7&2\\3&-3&1\\2&-1&0\end{bmatrix}$$

Question:2 (a) Given three points P(1, -1, -1), Q(1, 1, -1) and R(2, -1, 1), find
[8+8]

i. a unit vector perpendicular to the plane determined by P, Q and R.

ii. the area of the triangle PQR.

iii. the distance from R to the line through P and Q.

(b) Let A(0, 1, 2), B(1, -1, 0), C(1, 1, 2) and D(0, 1, 1) be a four points.

i. Find an equation of the plane determined by the points A, B, and C.

ii. Find the distance of the point D from the plane P: $y - z + 1 = 0$.

iii. Find parametric equations for the line L through D that is orthogonal to the plane $y - z + 1 = 0$..

Solution: $\vec{PQ} = \langle 0, 2, 0 \rangle$, $\vec{PR} = \langle 1, 0, 2 \rangle$

(a)

(i) Vector normal is $\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$

$= 4i - 2k$

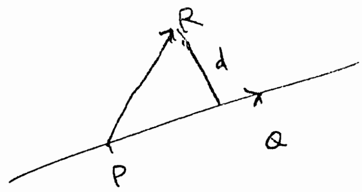
$$u = \frac{\vec{PQ} \times \vec{PR}}{\|\vec{PQ} \times \vec{PR}\|} = \frac{1}{\sqrt{16+4}}(4i - 2k) = \frac{4}{\sqrt{20}} i - \frac{2}{\sqrt{20}} k$$

$$= \frac{2}{\sqrt{5}} i - \frac{1}{\sqrt{5}} k$$

(ii) Area of $\triangle PQR$

$$\text{Area} = \frac{1}{2} \|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\sqrt{20} = \sqrt{5} \text{ unit}^2$$

(iii)



$$d = \frac{\|\vec{PQ} \times \vec{PR}\|}{\|\vec{PQ}\|}$$

$$= \frac{\sqrt{16+4}}{\sqrt{4}} = \sqrt{\frac{20}{4}} = \sqrt{5} \text{ unit}$$

(b) (i) $\vec{AB} = \langle 1, -2, -2 \rangle$, $\vec{AC} = \langle 1, 0, 0 \rangle$

vector normal to plane $\hat{n} = \begin{vmatrix} i & j & k \\ 1 & -2 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 0i - 2j + 2k$

Let A (0, 1, 2) be the point

Equation of plane is $0(x-0) - 2(y-1) + 2(z-2) = 0$

$-2y + 2z - 2 = 0$

$y - z + 1 = 0$

(ii)

$$d = \frac{|1 - 1 + 1|}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

(iii) Point is (0, 1, 1) vector parallel to line is $n = \langle 0, -2, 2 \rangle$

Parametric equations of line are

$x = 0$

$y = 1 - 2t$

$z = 1 + 2t$, $t \in \mathbb{R}$.

Question: 3. (a) If the acceleration of moving particle along a curve C is given by

$$a(t) = -3\cos t\, i - 3\sin t\, j + 2k.$$

[8+10+8]

Find the vector valued function that determines the curve C given that initial velocity is $v(0) = 3j$ and initial position is $r(0) = 3i$.

(b) The position vector of a moving particle at time t is $r(t) = \langle t^3, t^2, t \rangle$.

Find the tangential and normal components of acceleration at time t. Also find curvature.

(c) Let $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$, use differentials to approximate the change in $f(x,y,z)$, when the point (x,y,z) moves from (4,3,0) to (4.2, 2.9, 0.2).

Solution (a)

$$a(t) = r''(t) = -3\cos t\, i - 3\sin t\, j + 2k$$

$$v(t) = \int (-3\cos t\, i - 3\sin t\, j + 2k)\, dt$$

$$= -3\sin t\, i + 3\cos t\, j + 2t\, k + C \quad , \quad C = c_1 i + c_2 j + c_3 k$$

$$v(0) = 3j \Rightarrow 3j = 3j + C \Rightarrow C = 0$$

$$r(t) = \int (-3\sin t\, i + 3\cos t\, j + 2t\, k)\, dt$$

$$= 3\cos t\, i + 3\sin t\, j + t^2 k + C \quad C = c_1 i + c_2 j + c_3 k$$

$$r(0) = 3i \Rightarrow 3i = 3i + C \Rightarrow C = 0$$

$$r(t) = 3\cos t\, i + 3\sin t\, j + t^2 k.$$

(b)

$$r(t) = \langle t^3, t^2, t \rangle$$

$$r'(t) = \langle 3t^2, 2t, 1 \rangle$$

$$r''(t) = \langle 6t, 2, 0 \rangle.$$

$$r'(t) \cdot r''(t) = 18t^3 + 4t$$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 3t^2 & 2t & 1 \\ 6t & 2 & 0 \end{vmatrix} = -2i + 6t\, j - 6t^2 k$$

$$\|r'(t)\| = \sqrt{9t^4 + 4t^2 + 1}$$

$$\|r' \times r''\| = \sqrt{4 + 36t^2 + 36t^4}$$

$$a_T = \frac{r' \cdot r''}{\|r'\|} = \frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}}, \quad a_N = \frac{\sqrt{4 + 36t^2 + 36t^4}}{\sqrt{9t^4 + 4t^2 + 1}} \quad \frac{\|r' \times r''\|}{\|r'\|}$$

$$K = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{\sqrt{4 + 36t^2 + 36t^4}}{\left[\sqrt{(9t^4 + 4t^2 + 1)}\right]^3}$$

(c) $x = 4, \quad y = 3, \quad z = 0, \quad dx = \Delta x = 0.2, \quad dy = \Delta y = -0.1, \quad dz = \Delta z = 0.2$

$$df = f_x\, dx + f_y\, dy + f_z\, dz$$

$$df = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\, dx + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\, dy + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\, dz$$

$$df(4,3,0) = \frac{4}{5}(0.2) + \frac{3}{5}(-.01) + 0 = \frac{0.5}{5} = 0.1 \text{ unit}$$

Question: 4. (a) Use Chain rule to find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ where $w = r^2 + s^2$, $r = x - y$ and $s = x + y$.

[6+6+10] (b) Find the points on the paraboloid $z = x^2 + y^2$ at which the the normal line is parallel to the line through the points A(1,-1, 0) and B(0, 1, 1)

(c) Find the directional derivative of $f(x, y, z) = xze^y + \cos(xy)$ at the point P (2,0,1) In the direction of the line $x = -1 + 3t,\ y = 2 - 4t,\ z = 1 - 5t$

In which direction it increases most rapidly? What is the maximum rate of increase of f at P?

Solution

(a) $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$= 2r\cdot(1) + (2s)\cdot(1) = 2r + 2s = 2(x-y) + 2(x+y) = 2x - 2y + 2x + 2y = 4x$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= 2r\cdot(-1) + (2s)(1) = -2r + 2s = -2(x-y) + 2(x+y) = -2x + 2y + 2x + 2y = 4y$$

(b) $F(x,y,z) = x^2 + y^2 - z = 0$

$\nabla F = \langle 2x, 2y, -1\rangle$

$\overrightarrow{AB} = \langle -1, 2, 1\rangle$

∇F is parallel to $\overrightarrow{AB}$ $\Rightarrow$ $\frac{2x}{-1} = \frac{2y}{2} = \frac{-1}{1}$

$\Rightarrow$ $x = \frac{1}{2}$, $y = -1$

$z = x^2 + y^2 = \frac{1}{4} + 1 = \frac{5}{4}$

Required point is $\left(\frac{1}{2}, -1, \frac{5}{4}\right)$.

(c) $\nabla f = \langle ze^y - y\sin(xy),\ xze^y - x\sin(xy),\ xe^y\rangle$

$\nabla f(2,0,1) = \langle 1, 2, 2\rangle$

In Direction of vector $a = \langle 3, -4, 5\rangle$

$\|a\| = \sqrt{9+16+25} = \sqrt{50}$

$u = \frac{1}{\sqrt{50}}\langle 3, -4, 5\rangle$

- $D_u f(2,0,1) = \langle 1, 2, 2\rangle \cdot \frac{1}{\sqrt{50}}\langle 3, -4, 5\rangle$

$$= \frac{1}{\sqrt{50}}(3 - 8 + 10) = \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

- $f(2,0,1)$ increases most rapidly in direction $\nabla f(2,0,1) = \langle 1, 2, 2\rangle$.
- Maximum rate of increase is $\|\nabla f\| = \sqrt{1+4+4} = \sqrt{9} = 3$.

Question: 5. (a) Find local extrema and saddle points , if any, on the surface

[8+8] $$f(x,y) = x^3 - y^2 - 3x + 2y.$$

(b) Use method of Langrange multiplier to find extrema of the function $f(x,y) = x + 2y$, subject to constraint $x^2 + y^2 = 1$

Soln. (a)

$f_x = 3x^2 - 3$ $\qquad f_{xx} = 6x$

$f_y = -2y + 2$ $\qquad f_{xy} = 0$

$f_{yy} = -2$

critical points.

$3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$-2y + 2 = 0 \Rightarrow y = 1$

Points are $(1, 1), (-1, 1)$

Discreminant. $D(x,y) = f_{xx} f_{yy} - f_{xy}^2$

$= (6x)(-2) - 0 = -12x$

At point $(1, 1)$

$D(1, 1) = -12 < 0 \Rightarrow$ saddle point.

$f(1, 1) = -1$

At point $(-1, 1)$, $D(-1, 1) = 12 > 0$ relative extrema

$f_{yy}(-1, 1) = -2 < 0 \Rightarrow$ relative Maxima

$f(-1, 1) = 3.$

(b) $f(x,y) = x + 2y$, $\qquad g(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = \langle 1, 2 \rangle$, $\qquad \nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \Rightarrow$ $1 = 2\lambda x$ $\Rightarrow$ $\lambda = \frac{1}{2x}$

$2 = 2\lambda y$ $\qquad 2 = 2 \cdot \frac{1}{2x} y$

$x^2 + y^2 = 1$ $\qquad y = 2x$

$x^2 + 4x^2 = 1 \Rightarrow 5x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{5}}$

$y = \pm \frac{2}{\sqrt{5}}$

Points are $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{-1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

$f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \sqrt{5} \longrightarrow$ Maximum

$f\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) = -\frac{3}{\sqrt{5}}$

$f\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{3}{\sqrt{5}}$

$f\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) = -\sqrt{5} \longrightarrow$ Minimum.