TIME: 3 Hours **KING SAUD UNIVERSITY** **FULL MARKS: 80**

M - 107 **DEPARTMENT OF MATHEMATICS**

FINAL EXAMINATION (FIRST SEMESTER 1434-1435)

Question: 1: (a) Use Cramer's rule to solve the system

[6+5+3]

$$\begin{aligned} x + 2y + 3z &= 17 \\ 3x + 2y + z &= 11 \\ x - 5y + z &= -5 \end{aligned}$$

(b) Find the matrix B if $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$ **and** $AB = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

(c) If C is a 4x4 matrix with det C = 6, find

$i)\det C^{-1}$, ii) $\det C^{T}$, $iii)\det 3C$.

Question:2 (a) Given three points $P(1,-1,0)$, $Q(2,1,1)$ and $R(-1,1,2)$,

[6+8]

i. **Find a unit vector perpendicular to the plane determined by P, Q and R.**

ii. **Find the area of the triangle PQR.**

iii. **Find the component of** $\overrightarrow{PQ}$ **along** $\overrightarrow{PR}$ **and the vector projection of** $\overrightarrow{PQ}$ **onto** $\overrightarrow{PR}$.

(b) The position vector of a moving particle at time t is given by $r(t) = e^t \sin t\ i + e^t \cos t\ j + t\ k$. **Find the tangential and the normal components of acceleration and the curvature at** $t = \frac{\pi}{2}$.

Question: 3. (a) For the surface $36x^2 - 16y^2 - 9z^2 = 0$

[6+6+6]

(i) Write the name of the surface,

(ii) Write the names and the equations of the traces of the surface on the co-ordinate planes, and Sketch the surface.

(b) Show that $\lim_{(x,y)\to(0,0)} \frac{xy + x^2}{x^2 + y^2}$ **does not exist.**

(c) The dimensions of a closed rectangular box are measured as x =30 centimeters, y =20 centimeters and z =14 centimeters , with possible error of ± 0.01 **centimeters. Use differential to approximate the maximum error in calculated value of the volume of the box.**

Question: 4. (a) Use chain rule to find $\frac{\partial w}{\partial \rho}, \frac{\partial w}{\partial \phi}$ and $\frac{\partial w}{\partial \theta}$

[8+6+8]

where $w = xyz$, and $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$.

(b) Find equations of the tangent plane and the normal line to the surface $z = \sqrt{4 - x^2 - 2y^2}$ **at the point (1, -1, 1)**

(c) Find the directional derivative of $f(x,y,z) = x^2 e^{yz}$ **at the point P** (2,3,0) **in the direction of the line** $x = -1 + t,\ y = 2 - t,\ z = 1 - 2t$,
In which direction at P the function f **increases most rapidly ? What is the maximum rate of increase of** f **at P?**

Question: 5. (a) If $f(x,y) = x^2 + y^2 + x^2 y + 1$, **find local extrema and saddle points of** $f(x,y)$

[6+6]

(b) If $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$, **use Lagrange multipliers to find points on the plane** $x + y + z = 1$ ***at*** **which** $f(x,y,z)$ **has minimum value.**

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Question: 1: (a) Use Cramer's rule to solve the system

[6+5+3]

$$x + 2y + 3z = 17$$
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$$x - 5y + z = -5$$

(b) Find the matrix B if $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$ **and** $AB = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

(c) If C is a 4x4 matrix with det C = 6, find

$i)\det C^{-1}$, ii) $\det C^{T}$, $iii)\det 3C$.

Solution. (a)

(1) $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix} = -48$

(1) $\det A_1 = \begin{vmatrix} 17 & 2 & 3 \\ 11 & 2 & 1 \\ -5 & -5 & 1 \end{vmatrix} = -48$

(1) $\det A_2 = \begin{vmatrix} 1 & 17 & 3 \\ 3 & 11 & 1 \\ 1 & -5 & 1 \end{vmatrix} = -96$

(1) $\det A_3 = \begin{vmatrix} 1 & 2 & 17 \\ 3 & 2 & 11 \\ 1 & -5 & -5 \end{vmatrix} = -192$

(2) $x = \dfrac{\det A_1}{\det A} = \dfrac{-48}{-48} = 1$, $y = \dfrac{\det A_2}{\det A} = \dfrac{-96}{-48} = 2$, $z = \dfrac{\det A_3}{\det A} = \dfrac{-192}{-48} = 4$.

(b) $B = A^{-1}AB$

(3) $A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$, $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$

(2) $B = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 & -4 & 2 \\ 9 & 8 & -2 \end{bmatrix}$

(c) C is 4×4, $\det C = 6$

(1) (i) $\det C^{-1} = \dfrac{1}{\det C} = \dfrac{1}{6}$

(1) (ii) $\det C^{T} = \det C = 6$

(1) (iii) $\det 3C = 3^4 \det C = 81 \times 6 = 486$

Question:2 (a) Given three points $P(1,-1,0)$, $Q(2,1,1)$ and $R(-1,1,2)$,

[6+8]

i. **Find a unit vector perpendicular to the plane determined by P, Q and R.**
ii. **Find the area of the triangle PQR.**
iii. **Find the component of PQ along PR and the vector projection of PQ onto PR.**

(b) The position vector of a moving particle at time t is given by $r(t)=e^t \sin t\ i + e^t \cos t\ j + t\ k$. Find the tangential and the normal components of acceleration and the curvature at $t=\frac{\pi}{2}$.

Solution.

(a) $\vec{PQ} = \langle 1, 2, 1\rangle, \quad \vec{PR} = \langle -2, 2, 2\rangle$

(i) Vector perpendicular to the plane determined by P, Q and R is

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ -2 & 2 & 2 \end{vmatrix} = 2i - 4j + 6k$$

$$\text{Unit vector} = \frac{\vec{PQ} \times \vec{PR}}{\|\vec{PQ} \times \vec{PR}\|}$$

$$\|\vec{PQ} \times \vec{PR}\| = \sqrt{4+16+36} = \sqrt{56}$$

3 $\quad u = \frac{1}{\sqrt{56}} \langle 2, -4, 6\rangle$

1 (ii) Area of $\triangle PQR = \frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\sqrt{56} = \frac{1}{2} \times 2\sqrt{14} = \sqrt{14}$ unit^2

1 (iii) $\text{Comp}_{\vec{PR}} \vec{PQ} = \frac{\vec{PQ} \cdot \vec{PR}}{\|\vec{PR}\|} = \frac{-2+4+2}{\sqrt{4+4+4}} = \frac{4}{\sqrt{12}} = \frac{2}{\sqrt{3}}$

1 $\quad \text{Proj}_{\vec{PR}} \vec{PQ} = \left[\frac{\vec{PQ} \cdot \vec{PR}}{\|\vec{PR}\|}\right] \cdot \frac{\vec{PR}}{\|PR\|} = \frac{2}{\sqrt{3}} \cdot \frac{\langle -2, 2, 2\rangle}{\sqrt{12}} = \frac{1}{3}\langle -2, 2, 2\rangle$

(b) $r(t) = e^t \sin t\ i + e^t \cos t\ j + t k$

2 $\quad r'(t) = (e^t \sin t + e^t \cos t)\ i + (e^t \cos t - e^t \sin t)\ j + k$

$\quad r''(t) = 2e^t \cos t\ i - 2e^t \sin t\ j$

At $t=\frac{\pi}{2}$ $\quad r'(\frac{\pi}{2}) = \langle e^{\frac{\pi}{2}}, -e^{\frac{\pi}{2}}, 1\rangle, \quad \|r'(\frac{\pi}{2})\| = \sqrt{2e^{\pi}+1}$

2 $\quad r''(\frac{\pi}{2}) = \langle 0, -2e^{\frac{\pi}{2}}, 0\rangle$

$\quad r'(\frac{\pi}{2}) \cdot r''(\frac{\pi}{2}) = 2e^{\pi}, \quad r'(\frac{\pi}{2}) \times r''(\frac{\pi}{2}) = \langle 2e^{\frac{\pi}{2}}, 0, -2e^{\pi}\rangle$ 1

$\quad \|r'(\frac{\pi}{2}) \times r''(\frac{\pi}{2})\| = 2\sqrt{e^{\pi}+e^{2\pi}}$

1 $\quad a_T = \frac{r' \cdot r''}{\|r'\|} = \frac{2e^{\pi}}{\sqrt{2e^{\pi}+1}}$

1 $\quad K = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{2\sqrt{e^{\pi}+e^{2\pi}}}{[2e^{\pi}+1]^{3/2}}$

1 $\quad a_N = \frac{\|r' \times r''\|}{\|r'\|} = \frac{2\sqrt{e^{\pi}+e^{2\pi}}}{\sqrt{2e^{\pi}+1}}$

Question: 3. (a) For the surface $36x^2 - 16y^2 - 9z^2 = 0$

[6+6+6]

(i) Write the name of the surface,

(ii) Write the names and the equations of the traces of the surface on the co-ordinate planes, and Sketch the surface.

(b) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy + x^2}{x^2 + y^2}$ does not exist.

(c) The dimensions of a closed rectangular box are measured as x =30 centimeters, y =20 centimeters and z =14 centimeters , with possible error of ± 0.01 centimeters. Use differential to approximate the maximum error in calculated value of the volume of the box.

Solution (a) $36x^2 - 16y^2 - 9z^2 = 0$

OR $36x^2 = 16y^2 + 9z^2$

$$\frac{36x^2}{144} = \frac{16y^2}{144} + \frac{9z^2}{144}$$

1 (i) $\dfrac{x^2}{4} = \dfrac{y^2}{9} + \dfrac{z^2}{16}$ is a cone.

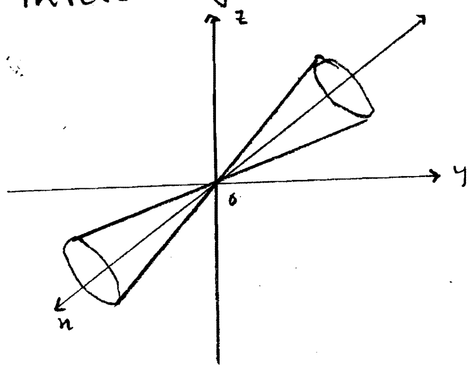
(ii)

Co-ord. Plane	Equation	Name
$x=0$, yz-plane	$\frac{y^2}{9} + \frac{z^2}{16} = 0$	Point $(0,0,0)$
$y=0$, xz-plane	$\frac{x^2}{4} = \frac{z^2}{16}$, $z = \pm 2x$	intersecting lines
$z=0$, xy-plane	$\frac{x^2}{4} = \frac{y^2}{9}$, $y = \pm \frac{3}{2}x$	intersecting line

3

Sketch

2



(b)

Path 1 Along x-axis, $y=0$

2 $\lim\limits_{(x,0)\to(0,0)} \dfrac{0+x^2}{x^2+0} = \lim\limits_{(x,0)\to(0,0)} \dfrac{x^2}{x^2} = \lim\limits_{(x,0)\to(0,0)} 1 = 1 \quad \to (1)$

Path 2 Along y-axis, $x=0$

2 $\lim\limits_{(0,y)\to(0,0)} \dfrac{0}{y^2} = \lim\limits_{(0,y)\to(0,0)} 0 = 0 \quad \to (2)$

$1 \neq 2 \Rightarrow$ Limit does not exist.

2

3.(c) $V = xyz$

(2) $dV = yz\,dx + xz\,dy + xy\,dz$

(1) $x = 30,\ y = 20,\ z = 15,\ dx = dy = dz = \pm 0.01$

(3) $dV = (300)(\pm 0.01) + (450)(\pm 0.01) + (600)(\pm 0.01)$

$= (1350)(\pm 0.01) = 13.5\ cm^3$

Question: 4. (a) Use chain rule to find $\frac{\partial w}{\partial \rho}, \frac{\partial w}{\partial \phi}$ and $\frac{\partial w}{\partial \theta}$. 5

[8+6+8] where $w = xyz$, and $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$.

(b) Find equations of the tangent plane and the normal line to the surface $z = \sqrt{4 - x^2 - 2y^2}$ at the point (1, -1, 1)

(c) Find the directional derivative of $f(x, y, z) = x^2 e^{yz}$ at the point $P(2,3,0)$ in the direction of the line $x = -1 + t,\ y = 2 - t,\ z = 1 - 2t$,

In which direction at P the function f increases most rapidly ? What is the maximum rate of increase of f at P?

Solution. (a) $w = xyz$, where $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \rho}$$

$$= yz \cdot \sin\phi\cos\theta + xz\sin\phi\sin\theta + xy\cos\phi$$

8

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \phi}$$

$$= yz \cdot \rho\cos\phi\cos\theta + xz\rho\cos\phi\sin\theta - xy\rho\sin\phi$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$= -yz\rho\sin\phi\sin\theta + xz\rho\sin\phi\cos\theta$$

(b) $f(x,y,z) = \sqrt{4 - x^2 - 2y^2} - z = 0$

2 $f_x = \frac{-x}{\sqrt{4-x^2-2y^2}}$, $f_y = \frac{-2y}{\sqrt{4-x^2-2y^2}}$, $f_z = -1$

At $(1,-1,1)$ $f_x = -1$, $f_y = 2$, $f_z = -1$

2 $N = \nabla f(1,-1,1) = -i + 2j - k$ is normal to surface at $(1,-1,1)$

1 Equation of tangent plane $-(x-1) + 2(y+1) - (z-1) = 0$

1 Equation of Normal line
$x = 1 - t$
$y = -1 + 2t$ $t \in \mathbb{R}$.
$z = 1 - t$

2 (c) $\nabla f(x,y,z) = \langle 2xe^{yz}, x^2ze^{yz}, x^2ye^{yz}\rangle$
$\nabla f(2,3,0) = \langle 4, 0, 12\rangle$

vector in direction of line $a = \langle 1,-1,-2\rangle$

1 unit vector $u = \frac{a}{\|a\|} = \langle \frac{1}{\sqrt6}, \frac{-1}{\sqrt6}, \frac{-2}{\sqrt6}\rangle$

2 • $D_u f(2,3,0) = \nabla f \cdot u = \frac{4}{\sqrt6} - \frac{24}{\sqrt6} = -\frac{20}{\sqrt6}$

2 • Direction of increase most rapidly $= \nabla f(2,3,0) = \langle 4, 0, 12\rangle$

1 • maximum rate of increase $= \|\nabla f\| = \sqrt{16 + 144} = \sqrt{208}$

Question: 5. (a) **If** $f(x,y)=x^2+y^2+x^2y+1$, **find local extrema and saddle points of** $f(x,y)$

[6+6] **(b) If** $f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2$, **use Lagrange multipliers to find points on the plane** $x+y+z=1$ *at* **which** $f(x,y,z)$ **has minimum value.**

Solution (a) $f_x = 2x + 2xy = 0 \Rightarrow 2x(1+y) = 0$

$x=0,\ y=-1$

$f_y = 2y + x^2 = 0 \qquad y = -\frac{x^2}{2}$

$x=0,\ y=0$

$y=-1 \quad x^2=2 \Rightarrow x = \mp\sqrt{2}$

3 critical points are $(0,0),\ (\sqrt{2},-1),\ (-\sqrt{2},-1)$

$f_{xx} = 2+2y,\quad f_{xy} = 2x,\quad f_{yy} = 2$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4+4y-4x^2$

1 At $(0,0)$ $\quad D(0,0)=4>0$ and $f_{yy}=2>0 \Rightarrow$ local mimum.

$f(0,0)=1$

1 At $(\sqrt{2},0)$ $\quad D(\sqrt{2},-1) = 4-4-8 = -8<0 \Rightarrow$ saddle point

1 At $(-\sqrt{2},0)$ $\quad D(-\sqrt{2},-1) = 4-4-8 = -8<0 \Rightarrow$ saddle point

(b) $f(x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2,\quad g(x,y,z) = x+y+z-1=0$

2 $\nabla f = \langle 2(x-1), 2(y-1), 2(z-1)\rangle,\quad \nabla g = \langle 1,1,1\rangle$

$\nabla f = \lambda \nabla g$

$\Rightarrow$

$2(x-1)=\lambda \qquad 2x = \lambda+2, \quad x = \frac{\lambda}{2}+1$

$2(y-1)=\lambda \qquad 2y = \lambda+2 \quad y = \frac{\lambda}{2}+1$

2 $2(z-1)=\lambda \qquad 2z = \lambda+2 \quad z = \frac{\lambda}{2}+1$

$g(x,y,z) = x+y+z-1=0$

$\frac{\lambda}{2}+1\ \ \frac{\lambda}{2}+1 + \frac{\lambda}{2}+1 = 1 \qquad \frac{3\lambda}{2} = 1-3 = -2$

$\lambda = -\frac{4}{3}$ 1

$x = -\frac{2}{3}+1 = \frac{1}{3},\quad y=\frac{1}{3},\quad z=\frac{1}{3}$

1 $f(x,y,z)$ is minimum at point $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.