King Saud University
College of Science
Department of Mathematics

M 106 - INTEGRAL CALCULUS

Solutions of the second midterm exam
Second Semester 1432-1433 H

Dr. Tariq A. AlFadhel[1][2]

[1] E-mail : alfadhel@ksu.edu.sa
[2] URL : http://faculty.ksu.edu.sa/alfadhel

Multiple choice questions (One mark for each question)

Question 1. $\lim_{x\to 0} \dfrac{2^x - 3^x}{x}$ is equal to

(a) ∞

(b) $\ln \frac{2}{3}$

(c) $\ln \frac{3}{2}$

(d) -1

Answer: $\lim_{x\to 0} \dfrac{2^x - 3^x}{x} \qquad \left(\dfrac{0}{0}\right)$

Apply L'Hôpital's rule

$\lim_{x\to 0} \dfrac{2^x - 3^x}{x} = \lim_{x\to 0} \dfrac{2^x \ln 2 - 3^x \ln 3}{1} = 2^0 \ln 2 - 3^0 \ln 3 = \ln 2 - \ln 3 = \ln \dfrac{2}{3}$

The right answer is (b)

Question 2. The partial fraction decomposition of $\dfrac{x^2+2}{(x^4-1)(x-1)}$

(a) $\dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2-1} + \dfrac{E}{x-1}$

(b) $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} + \dfrac{E}{x+1}$

(c) $\dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x-1)^2} + \dfrac{E}{x+1}$

(d) $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{(x-1)^2} + \dfrac{E}{x+1}$

Answer: $\dfrac{x^2+2}{(x^4-1)(x-1)} = \dfrac{x^2+2}{(x^2+1)(x^2-1)(x-1)} = \dfrac{x^2+2}{(x^2+1)(x-1)^2(x+1)}$

$\dfrac{x^2+2}{(x^4-1)(x-1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} + \dfrac{E}{x+1}$

The right answer is (b)

Question 3. The integral $\displaystyle\int \frac{2\ dx}{x^2-4x+3}$ is equal to

(a) $\ln|x^2-4x+3| + c$

(b) $\ln\left|\dfrac{x+1}{x+3}\right| + c$

(c) $\ln\left|\dfrac{x-3}{x-1}\right| + c$

(d) $\ln\left|\dfrac{x-1}{x-3}\right| + c$

Answer : Using partial fractions

$$\frac{2}{x^2-4x+3}=\frac{2}{(x-1)(x-3)}=\frac{A}{x-1}+\frac{B}{x-3}$$

$2=A(x-3)+B(x-1)$

Put $x=1$ then $2=A(1-3)\Rightarrow 2=-2A\Rightarrow A=-1$

Put $x=3$ then $2=2B\Rightarrow B=1$

$$\int\frac{2}{x^2-4x+3}\,dx=\int\left(\frac{-1}{x-1}+\frac{1}{x-3}\right)\,dx$$

$$\int\frac{2}{x^2-4x+3}\,dx=-\ln|x-1|+\ln|x-3|+c=\ln\left|\frac{x-3}{x-1}\right|+c$$

The right answer (c)

Question 4. The value of the integral $\int\sin^5 x\cos^3 x\,dx$ is equal to

(a) $\frac{1}{6}\sin^6 x-\frac{1}{8}\sin^8 x+c$

(b) $\frac{1}{5}\sin^5 x-\frac{1}{3}\sin^3 x+c$

(c) $\frac{1}{3}\sin^5 x-\frac{1}{2}\sin^2 x+c$

(d) $\frac{1}{3}\sin^5 x-\frac{1}{8}\sin^8 x+c$

Answer : $\int\sin^5 x\cos^3 x\,dx=\int\sin^5 x\cos^2 x\cos x\,dx$

$=\int\sin^5 x(1-\sin^2 x)\cos x\,dx$

Put $u=\sin x\Rightarrow du=\cos x\,dx$

$\int\sin^5 x(1-\sin^2 x)\cos x\,dx=\int u^5(1-u^2)\,du$

$$=\int(u^5-u^7)\,du=\frac{u^6}{6}-\frac{u^8}{8}+c$$

$$=\frac{1}{6}\sin^6 x-\frac{1}{8}\sin^8 x+c$$

The right answer is (a)

Question 5. The substitution $u=\tan\left(\frac{x}{2}\right)$ transforms the integral $\int\frac{1}{3-\sin x+2\cos x}\,dx$ into

(a) $\int\frac{2}{u^2-2u+5}\,du$

(b) $\int\frac{2}{u^2-2u+3}\,du$

(c) $\int\frac{2}{u^2+2u+5}\,du$

(d) $\int\frac{2}{u^2+2u+3}\,du$

Answer : Using the half angle substitution $u=\tan\left(\frac{x}{2}\right)$

$$\cos x=\frac{1-u^2}{1+u^2}\ ,\ \sin x=\frac{2u}{1+u^2}\ \text{and}\ dx=\frac{2}{1+u^2}\,du$$

$$\int \frac{1}{3-\sin x+2\cos x}\,dx = \int \frac{1}{3-\frac{2u}{1+u^2}+2\left(\frac{1-u^2}{1+u^2}\right)}\,\frac{2}{1+u^2}\,du$$
$$= \int \frac{1+u^2}{3(1+u^2)-2u+2(1-u^2)}\,\frac{2}{1+u^2}\,du = \int \frac{2}{3+3u^2-2u+2-2u^2}\,du$$
$$= \int \frac{2}{u^2-2u+5}\,du$$
The right answer is (a)

Question 6. To evaluate the integral $\int \frac{dx}{x^2\sqrt{x^2-25}}$, we use the substitution

(a) $x = 5\sec\theta$

(b) $x = \sec^5\theta$

(c) $x = 5\tan\theta$

(d) $x = \tan^5\theta$

Answer : $\sqrt{x^2-25} = \sqrt{(x)^2-(5)^2}$
So, we use the substitution $x = 5\sec\theta$
The right answer is (a)

Question 7. The improper integral $\int_0^\infty \frac{1}{x^2+1}\,dx$

(a) converges to π

(b) diverges

(c) converges to $\frac{\pi}{2}$

(d) converges to $+\infty$

Answer : $\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+1}\,dx$
$= \lim_{t\to\infty} \left[\tan^{-1}(x)\right]_0^t = \lim_{t\to\infty} \left[\tan^{-1}(t) - \tan^{-1}(0)\right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$
The right answer is (c)

Question 8. The value of the integral $\int \frac{1}{1+e^x}\,dx$ is equal to

(a) $x - \ln|x+1| + c$

(b) $x - \ln(e^x+1) + c$

(c) $\frac{x^2}{2} - \ln(e^x+1) + c$

(d) $\ln\left(\frac{x^2}{2}\right) - \ln|x+1| + c$

Answer : $\displaystyle\int \frac{1}{1+e^x}\ dx = \int \frac{(1+e^x)-e^x}{1+e^x}\ dx$
$= \displaystyle\int \left(\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\right)\ dx = \int \left(1 - \frac{e^x}{1+e^x}\right)\ dx$
$= x - \ln(e^x+1) + c$
The right answer is (b)

Question 9. The value of the integral $\displaystyle\int \ln|x|\ dx$ is equal to

(a) $\dfrac{\ln|x|}{x} - x + c$

(b) $x\ln|x| - \ln|x| + c$

(c) $\dfrac{\ln^2|x|}{2} + c$

(d) $x\ln|x| - x + c$

Answer : Using integration by parts
$u = \ln|x| \qquad dv = dx$
$du = \dfrac{1}{x}\ dx \qquad v = x$
$\displaystyle\int \ln|x|\ dx = x\ln|x| - \int x\ \frac{1}{x}\ dx$
$= x\ln|x| - \int dx = x\ln|x| - x + c$
The right answer is (d).

Question 10. The area of the region bounded by the graphs of $y = x$, $y = -x$ and $y = 1$ is equal to

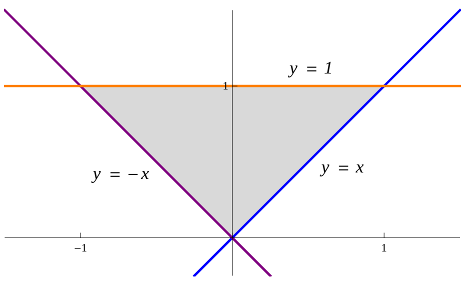
(a) 1

(b) 0

(c) 2

(d) $\frac{1}{2}$

Answer :



$y = x$ intersects $y = 1$ at $x = 1$ and $y = -x$ intersects $y = 1$ at $x = -1$
$\displaystyle Area = \int_{-1}^{0} (1-(-x))\ dx + \int_{0}^{1} (1-x)\ dx$

$$Area = \left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1}$$
$$Area = \left[0 - \left(-1 + \frac{1}{2}\right)\right] + \left[\left(1 - \frac{1}{2}\right) - 0\right] = \frac{1}{2} + \frac{1}{2} = 1$$
The right answer is (a)

Full questions

Question 11. Evaluate $\int \sin^5 5x \ dx$. [3 marks]

Answer : $\int \sin^5 5x \ dx = \int \sin^4 5x \ \sin 5x \ dx$
$= \int \left(\sin^2 5x\right)^2 \ \sin 5x \ dx = \int \left(1 - \cos^2 5x\right)^2 \sin 5x \ dx$
Put $u = \cos 5x$ then $du = -5 \sin 5x \ dx \Rightarrow -\frac{1}{5} \ du = \sin 5x \ dx$
$\int \left(1 - \cos^2 5x\right)^2 \sin 5x \ dx = -\frac{1}{5} \int \left(1 - u^2\right)^2 \ du$
$= -\frac{1}{5} \int (1 - 2u^2 + u^4) \ du = -\frac{1}{5}\left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right] + c$
$= -\frac{\cos 5x}{5} + \frac{2\cos^3 5x}{15} - \frac{\cos^5 5x}{25} + c$

Question 12. Evaluate the integral $\int \frac{x+1}{x^2 + x - 2} \ dx$ [2 marks]

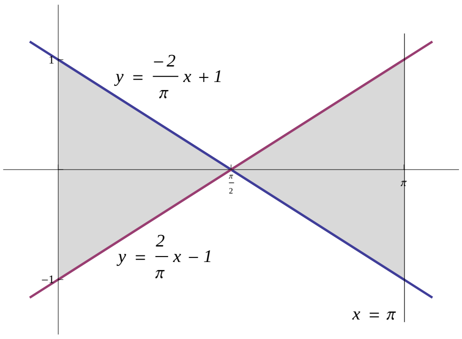
Answer : Using partial fractions
$\frac{x+1}{x^2+x-2} = \frac{x+1}{(x+2)(x-1)} = \frac{A}{x-1} + \frac{B}{x+2}$
$x + 1 = A(x+2) + B(x-1)$
Put $x = 1$ then $2 = 3A \Rightarrow A = \frac{2}{3}$
Put $x = -2$ then $-1 = -3B \Rightarrow B = \frac{1}{3}$
$\frac{x+1}{x^2+x-2} = \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}$
$\int \frac{x+1}{x^2+x-2} \ dx = \frac{2}{3} \int \frac{1}{x-1} \ dx + \frac{1}{3} \int \frac{1}{x+2} \ dx$
$= \frac{2}{3} \ln|x-1| + \frac{1}{3} \ln|x+2| + c$

Question 13. Evaluate the integral $\int \tan x \sec^3 x \ dx$ [3 marks]

Answer : $\int \tan x \sec^3 x \ dx = \int \sec^2 x \sec x \tan x \ dx$
$= \int (\sec x)^2 \ \sec x \tan x \ dx = \frac{\sec^3 x}{3} + c$

Question 14. **Sketch** the region bounded by the graphs of $y = -\frac{2}{\pi}x + 1$, $y = \frac{2}{\pi}x - 1$, $x = 0$ and $x = \pi$. Find its **area**. [2 marks]

Answer :
Points of intersection between $y = -\frac{2}{\pi}x + 1$ and $y = \frac{2}{\pi}x - 1$
$\frac{2}{\pi}x - 1 = -\frac{2}{\pi}x + 1 \Rightarrow \frac{4}{\pi}x = 2 \Rightarrow x = \frac{\pi}{2}$



$$Area = \int_0^{\frac{\pi}{2}} \left[-\frac{2}{\pi}x + 1 - \left(\frac{2}{\pi}x - 1\right)\right]\ dx + \int_{\frac{\pi}{2}}^{\pi} \left[\frac{2}{\pi}x - 1 - \left(-\frac{2}{\pi}x + 1\right)\right]\ dx$$

$$Area = \int_0^{\frac{\pi}{2}} \left(-\frac{4}{\pi}x + 2\right)\ dx + \int_{\frac{\pi}{2}}^{\pi} \left(\frac{4}{\pi}x - 2\right)\ dx$$

$$Area = \left[-\frac{2}{\pi}x^2 + 2x\right]_0^{\frac{\pi}{2}} + \left[\frac{2}{\pi}x^2 - 2x\right]_{\frac{\pi}{2}}^{\pi}$$

$$\underline{Area = \left[\left(-\frac{\pi}{2} + \pi\right) - 0\right] + \left[(2\pi - 2\pi) - \left(\frac{\pi}{2} - \pi\right)\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi}$$