KING SAUD UNIVERSITY
College of Science
Department of Mathematics

Second Semester (1434/1435)

Final Exam, M-106

Programmable Calculators are Not Authorized

Name:	Number:
Name of Teacher:	Group No:

Max Marks: 40 Time: Three hours

The Exam paper contains 8 pages
(**10 Multiple choice questions and 7 Full questions**)

Multiple Choice (1-10)	
Question # 11	
Question # 12	
Question # 13	
Question # 14	
Question # 15	
Question # 16	
Question # 17	
Total	

Multiple Choice

$Q.No:$	1	2	3	4	5	6	7	8	9	10
$\{a,b,c,d\}$	c	b	a	a	d	a	b	a	b	a

Q. No: 1 The integral $\int \frac{dx}{x^2+4}$ is equal to:

(a) $\frac{1}{2}\sec(\frac{x}{2})+C$ (b) $\frac{1}{2}\sec^{-1}(\frac{x}{2})+C$

(c) $\frac{1}{2}\tan^{-1}(\frac{x}{2})+C$ (d) $\frac{1}{2}\tan(\frac{x}{2})+C$

Q. No: 2 The graph of the curve $\mathcal{C}$ defined by the polar equations $r=2+2\sin(\theta)$, is:

(a) a Limacon (b) a cardiod (c) a rose (d) a circle

Q. No: 3 If a point has a polar coordinate $(r,\theta)=(2,\frac{\pi}{2})$ then its rectangle coordinate (x,y) is:

(a) $(0,2)$ (b) $(0,-2)$ (c) $(2,1)$ (d) $(2,0)$

Q. No: 4 The integral $\int \frac{\sinh(x)}{\cosh^2(x)}dx$ is equal to:

(a) $-\operatorname{sech}(x)+c$ (b) $-\frac{2}{\cosh(x)}+c$

(c) $\operatorname{sech}(x)+c$ (d) $\cosh(x)+c$

Q. No: 5 $\frac{d}{dx}[\int_2^{x^2} \ln(t^2)dt]$ is equal to:

(a) $2x\ln x$ (b) $x\ln x^2$ (c) $x^2\ln x^2$ (d) $2x\ln(x^4)$

Q. No: 6 $\lim_{x\to\infty}\frac{x+e^x}{1+e^{3x}}$ is equal to:

(a) 0 (b) ∞ (c) $\frac{1}{9}$ (d) 1

Q. No: 7 The integral $\int xe^x dx$ is equal to

(a) $\frac{x^2e^x}{2}+C$ (b) xe^x-e^x+C (c) $\frac{x^2e^x}{2}-e^x+C$ (d) $\frac{xe^x}{2}-e^x+C$

Q. No: 8 The slope of the tangent line to the parametric equations $x = 3\cos t$, $y = 3\sin t$, at $t = \frac{\pi}{4}$ is:

$(a)\ -1$ $(b)\ 1$ $(c)\ \frac{1}{\sqrt{2}}$ $(d)\ 1$

Q. No: 9 The average value of the function $f(x) = (x+1)^2$ on $[-2, 0]$ is equal to:

$(a)\ 0$ $(b)\ \frac{1}{3}$ $(c)\ \frac{2}{3}$ $(d)\ -\frac{2}{3}$

Q. No: 10 The surface area of the solid obtained by revolving the region bounded by $y = t$, $x = t$, $0 \leq t \leq 2$ about $x-$axis is equal to:

$(a)\ 4\sqrt{2}\pi$ $(b)\ 4\sqrt{2}$ $(c)\ 2\sqrt{2}\pi$ $(d)\ 4\pi$

Full Questions

x

Question No: 11 **Find** $y'(x)$ if $y = \sqrt{(x^2+1)\sqrt{x^2+1}}$

We have

$$\ln y(x) = \frac{1}{2}(\ln(x^2+1) + \frac{1}{2}\ln(x^2+1)) \ [1]$$

then

$$\frac{y'(x)}{y(x)} = \frac{1}{2}(\frac{2x}{x^2+1} + \frac{1}{2}\frac{2x}{x^2+1}) = \frac{1}{2}(\frac{3x}{x^2+1}) \ [1{+}1]$$

Then

$$y'(x) = \frac{3}{2}(\frac{x}{x^2+1})\sqrt{(x^2+1)\sqrt{x^2+1}} \ [1]$$

Question No: 12 **Evaluate** $\int \frac{1}{x^2+2x+3}dx$

Let

$$u = x+1, \ du = dx \ [1]$$

Then

$$\begin{aligned}\int \frac{1}{x^2+2x+3}dx &= \int \frac{1}{(x+1)^2+2}dx \ [1]\\ &= \int \frac{du}{u^2+2} \ [0.5]\\ &= \frac{1}{\sqrt{2}}\tan^{-1}(\frac{u}{\sqrt{2}}) + C \ \ [1]\\ &= \frac{1}{\sqrt{2}}\tan^{-1}(\frac{x+1}{\sqrt{2}}) + C \ [0.5]\end{aligned}$$

Question No: 13 **Find** $\lim_{x\to 0^+}(\frac{1}{\sin x})^{\sin x}$ if it exists.

Let

$$y(x) = \left(\frac{1}{\sin x}\right)^{\sin x}$$

then

$$\ln y(x) = \sin(x)\ln(\frac{1}{\sin x}) = \frac{-\ln(\sin(x))}{\frac{1}{\sin x}} \ [0.5]$$

And we get

$$\begin{aligned}\lim_{x\to 0^+}\frac{-\ln(\sin(x))}{\frac{1}{\sin x}} &= \lim_{x\to 0^+}\frac{-\ln(\sin(x))}{\frac{1}{\sin x}} \ [1]\\ &= \lim_{x\to 0^+}\frac{\frac{-\cos x}{\sin x}}{\frac{\cos x}{\sin^2 x}} \ [1]\\ &= \lim_{x\to 0^+} -\sin x = 0 \ [0.5]\end{aligned}$$

Finally we will have

$$\lim_{x\to 0^+}\left(\frac{1}{\sin x}\right)^{\sin x} = e^0 = 1 \ [1]$$

Question No: 14 **Evaluate** $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

Let

$$x = 2\tan\theta, \text{ we will have } dx = 2\sec^2\theta d\theta, \ [1]$$

and we get

$$\begin{aligned}\sqrt{x^2+4} &= \sqrt{4\tan^2(\theta)+4}\\ &= 2\sec\theta\end{aligned}$$

Its clear that

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+4}}dx &= \frac{1}{4}\int\frac{\sec\theta d\theta}{\tan^2(\theta)} \ [1]\\ &= \frac{1}{4}\int\frac{\cos(\theta)}{\sin^2(\theta)}d\theta \ [0.5]\\ &= \frac{1}{4}(\sin\theta)^{-1} + c \ [1]\\ &= \frac{1}{4}\frac{\sqrt{x^2+4}}{x} + c \ [0.5]\end{aligned}$$

Question No: 15 **Sketch,** and **Find** the length of the curve C given by $r = 2\sin(\theta)$.

Graph [1]

we have

$$\begin{aligned} L_0^\pi &= \int_0^\pi \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta \\ &= \int_0^\pi \sqrt{4\sin^2\theta + 4\cos^2\theta} d\theta \text{ [2]} \\ &= 2\int_0^\pi d\theta = 2\pi \text{ [1]} \end{aligned}$$

Question No: 16 Let C be the given by $x(t) = \sin(t) + 1$, $y(t) = \cos(t) - 1, 0 \le t \le \frac{\pi}{2}$. [5]

a) **Find** the equation of the curve in rectangle coordinate (x, y).

Let

$$\sin t = -1, \cos t = y + 1$$

Then

$$\begin{aligned} \sin^2(t) + \cos^2(t) &= 1 \\ (x-1)^2 + (y+1)^2 &= 1 \text{ [1]} \end{aligned}$$

b) **Set up** an integral that can be used to find the surface area generated by revolving C about $x-$axis.

$$\begin{aligned} S.A &= 2\pi \int_0^{\frac{\pi}{2}} |y| \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \\ &= 2\pi \int_0^{\frac{\pi}{2}} |\cos(t) - 1| \sqrt{(\cos t)^2 + (-\sin t)^2} dt \text{ [2]} \end{aligned}$$

c) **Find** the surface area generated by revolving C about $x-$axis.

$$\begin{aligned} S.A &= 2\pi \int_0^{\frac{\pi}{2}} |\cos(t) - 1| \, dt \\ &= 2\pi(\frac{1}{2}\pi - 1) \text{ [2]} \end{aligned}$$

Question No: 17 **Sketch** the region R bounded by $y = \sqrt{1-x^2}$ and the line $y = 0$ and **Find** the volume of the solid that is formed by revolving the region R about the $x-$axis.

Graph [1]

$$
\begin{aligned}
V &= \pi \int_{-1}^{1} (\sqrt{1-x^2})^2 dx \text{ (Disc Method) [2]} \\
&= \pi[\sqrt{1-x^2}]_{-1}^{1} \text{ [1]} \\
&= \frac{4}{3}\pi \text{ [1]}
\end{aligned}
$$

Other methods

$$
\begin{aligned}
V &= 4\pi \int_{0}^{1} y\sqrt{1-y^2} dy \text{ (Cylindrical shell) [2]} \\
&= -2\pi[\frac{(1-y^2)^{\frac{3}{2}}}{\frac{3}{2}}]_{-1}^{1} \text{ [1]} \\
&= \frac{4}{3}\pi \text{ [1]}
\end{aligned}
$$

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Second Semester 34-35

Multiple Choice

$Q1$: The integral $\int \frac{dx}{x^2+4}$ is equal to

(a) $\frac{1}{2}\sec\left(\frac{x}{2}\right)+c$ (b) $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)+c$ (c) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$ (d) $\frac{1}{2}\tan\left(\frac{x}{2}\right)+c$

solution

$$\int \frac{dx}{x^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c \qquad (c)$$

$Q2$: The graph of the curve C defined by the polar equation $r = 2+2\sin(\theta)$ is

(a) a Limacon (b) a cardiod (c) a rose (d) a circle

solution

a cardiod (b)

$Q3$: If a point has a polar coordinate $(r,\theta)=(2,\frac{\pi}{2})$ then its rectangle coordinate (x,y) is:

(a) $(0,2)$ (b) $(0,-2)$ (c) $(2,1)$ (d) $(2,0)$

solution:

$$x = r\cos(\theta) \longrightarrow x = 2\cos\left(\frac{\pi}{2}\right)=0$$

$$y = r\sin(\theta) \longrightarrow y = 2\sin\left(\frac{\pi}{2}\right)=2$$

$$(x,y)=(0,2) \qquad (a)$$

$Q4$: The integral $\int \frac{\sinh(x)}{\cosh^2(x)}dx$ is equal to :

(a) $-\text{sech}(x)+c$ (b) $-\frac{2}{\cosh(x)}+c$ (c) $\text{sech}(x)+c$ (d) $\cosh(x)+c$

solution:

$$\int \frac{\sinh(x)}{\cosh^2(x)}dx = \int \frac{\sinh(x)}{\cosh(x)}\cdot\frac{1}{\cosh(x)}dx$$

$$= \int \tanh(x)\,\text{sech}(x)\,dx$$

$$= -\text{sech}(x)+c \qquad (a)$$

محمد ندا (أبو يوسف) 0559 108 708

$Q5: \dfrac{d}{dx}\left[\int_{2}^{x^2} \ln(t^2)dt\right]$ is equal to

(a) $2x\ \ln(x)$ (b) $x\ \ln x^2$ (c) $x^2 \ln x^2$ (d) $2x\ \ln x^4$

solution:

$$\frac{d}{dx}\left[\int_{2}^{x^2} \ln(t^2)dt\right] = \ln\left((x^2)^2\right)\cdot 2x$$

$$= 2x\ \ln(x^4)$$ (d)

$Q6: \lim_{x\to\infty}\dfrac{x+e^x}{1+e^{3x}}$ is equal to

(a) 0 (b) ∞ (c) $\dfrac{1}{9}$ (d) 1

solution: is $\frac{\infty}{\infty}$

Using L'Hopital rule $= \lim_{x\to\infty}\dfrac{1+e^x}{3e^{3x}}$ is $\frac{\infty}{\infty}$

Using L'Hopital rule $= \lim_{x\to\infty}\dfrac{e^x}{9e^{3x}}$

$$= \lim_{x\to\infty}\frac{1}{9e^{2x}} = \frac{1}{\infty} = 0$$ (a)

$Q7:$ The integral $\int xe^x dx$ is equal to

(a) $\dfrac{x^2e^x}{2}+c$ (b) $xe^x - e^x + c$ (c) $\dfrac{x^2e^x}{2} - e^x + c$ (d) $\dfrac{xe^x}{2} - e^x + c$

solution:

by parts put $u = x$ $dv = e^x dx$

$du = dx$ $v = e^x$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$= xe^x - e^x + c$$ (b)

$Q8:$ The slope of tangent line to the parametric equations $x = 3\cos(t)$, $y = 3\sin(t)$ at $t = \dfrac{\pi}{4}$ is

(a) -1 (b) 1 (c) $\dfrac{1}{\sqrt{2}}$ (d) 1

solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos(t)}{-3\sin(t)}$$

Slope $= \dfrac{3\cos(\pi/4)}{-3\sin(\pi/4)} = -1$ (a)

$Q9:$ The average value of the function $f(x)=(x+1)^2$ on $[-2,0]$ is equal to

(a) 0 $\quad$ (b) $\frac{1}{3}$ $\quad$ (c) $\frac{2}{3}$ $\quad$ (d) $-\frac{2}{3}$

solution

$$f_{av}=\frac{1}{0-(-2)}\int_{-2}^{0}(x+1)^2dx$$

$$=\frac{1}{2}\left[\tfrac{1}{3}(x+1)^3\right]_{-2}^{0}$$

$$=\frac{1}{6}\left[(1)-(-1)\right]=\frac{1}{3}$$ (b)

$Q10:$ The surface area of the solid obtained by revolving the region bounded by

$y=t \quad , \quad x=t \quad , \quad 0\le t\le 2$ about $x-axis$ is equal to

(a) $4\sqrt{2}\pi$ $\quad$ (b) $4\sqrt{2}$ $\quad$ (c) $2\sqrt{2}\pi$ $\quad$ (d) 4π

solution:

$$\text{S.A}=\int_{0}^{2}2\pi\left|y(t)\right|\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt$$

$$=2\pi\int_{0}^{2}t\sqrt{(1)^2+(1)^2}\,dt$$

$$=2\pi\sqrt{2}\left[\tfrac{1}{2}t^2\right]_0^2=4\sqrt{2}\ \pi$$ (a)

Full Questions

$Q11:$ Find $y'(x)$ if $y=\sqrt{(x^2+1)\sqrt{x^2+1}}$

solution

$$\ln y=\ln\left((x^2+1)\sqrt{x^2+1}\right)^{\frac{1}{2}}$$

$$\ln y=\frac{1}{2}\left[\ln(x^2+1)+\ln(x^2+1)^{\frac{1}{2}}\right]$$

$$\ln y=\frac{1}{2}\left[\ln(x^2+1)+\frac{1}{2}\ln(x^2+1)\right]$$

$$\frac{y'}{y}=\frac{1}{2}\left[\frac{2x}{x^2+1}+\frac{1}{2}\cdot\frac{2x}{x^2+1}\right]=\frac{1}{2}\left[\frac{3x}{x^2+1}\right]$$

$$y'=\frac{1}{2}\left[\frac{3x}{x^2+1}\right]\sqrt{(x^2+1)\sqrt{x^2+1}}$$

$Q12:$ Evaluate $\int \frac{1}{x^2+2x+3}dx$

solution:

completing square : $x^2+2x+3 = x^2+2x+(1)^2-(1)^2+3$

$= (x+1)^2+2$

$$\int \frac{1}{x^2+2x+3}dx = \int \frac{1}{(x+1)^2+2}dx \qquad \text{Let } u = x+1 \quad , \quad du = dx$$

$$= \int \frac{1}{u^2+(\sqrt{2})^2}du$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

$Q13:$ Find $\lim_{x\to 0+}\left(\frac{1}{\sin x}\right)^{\sin x}$

solution: is ∞^0 form

Let $y = \left(\frac{1}{\sin x}\right)^{\sin x}$

$$\ln y = \ln\left(\frac{1}{\sin x}\right)^{\sin x}$$

$$= \sin x \cdot \ln\left(\frac{1}{\sin x}\right) = -\sin x \ln(\sin x)$$

$$* \ \ln\left(\frac{1}{\sin x}\right) = \ln(1) - \ln(\sin x) = -\ln(\sin x)$$

$$\lim_{x\to 0+}\ln y = \lim_{x\to 0+} -\sin x \ln(\sin x) \qquad \text{is } 0\cdot\infty \text{ form}$$

$$= \lim_{x\to 0+}\frac{-\ln(\sin x)}{\frac{1}{\sin x}} \qquad \text{is } \frac{0}{0} \text{ form}$$

Applying L'Hopital rule

$$= \lim_{x\to 0+}\frac{-\frac{\cos x}{\sin x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0+}\sin x = 0$$

we get $\lim_{x\to 0+}\ln y = 0 \longrightarrow \lim_{x\to 0+} y = e^0 = 1$

$$\lim_{x\to 0+}\left(\frac{1}{\sin x}\right)^{\sin x} = 1$$

0559 108 708 (محمد ندا (أبو يوسف

$Q14:$ Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

solution:

Let $x = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$

$$** \sqrt{x^2+4} = \sqrt{(2\tan\theta)^2+4}$$

$$= \sqrt{4\tan^2\theta+4}$$

$$= \sqrt{4(\tan^2\theta+1)}$$

$$= \sqrt{4\sec^2\theta} = 2\sec\theta$$

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta}\cdot 2\sec^2\theta d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta$$

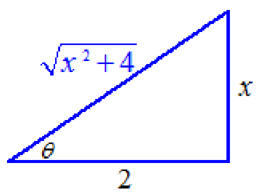
$$= \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta$$

$$= \frac{1}{4}\int (\sin\theta)^{-2}\cos\theta d\theta = \frac{1}{-4}(\sin\theta)^{-1}+c$$

$$= \frac{1}{4}\cdot\frac{1}{\sin\theta}+c$$

$$= \frac{1}{4}\cdot\frac{\sqrt{x^2+4}}{x}+c$$

$x = 2\tan\theta$, $\tan\theta = \frac{x}{2}$



$Q15:$ Sketch , and Find the length of the curve C given by $r = 2\sin(\theta)$

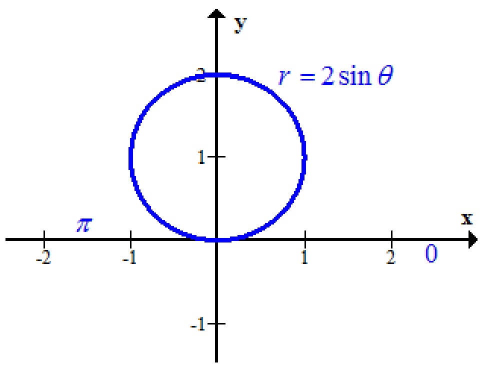
solution:

$$L_0^\pi = \int_0^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{(2\sin\theta)^2+(2\cos\theta)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{4(\sin^2\theta+\cos^2\theta)}\, d\theta$$

$$= \int_0^\pi 2d\theta = [2\theta]_0^\pi = 2\pi$$



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$Q16:$ Let C be the given by $x(t)=\sin(t)+1$, $y(t)=\cos(t)-1$, $0 \le t \le \frac{\pi}{2}$

a) Find the equation of the curve in rectangle coordinate (x, y)

b) Set up the integral that can be used to find the surface area generated by revolving C about $x-axis$

c) Find the surface area generated by revolving C about $x-axis$

solution:

a)

$x = \sin(t)+1 \longrightarrow \sin(t) = x-1$

$y = \cos(t)-1 \longrightarrow \cos(t) = y+1$

** $\sin^2(t) + \cos^2(t) = 1$

$(x-1)^2 + (y+1)^2 = 1$ a circle

$b)$

$$S.A = \int_0^{\frac{\pi}{2}} 2\pi |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} |\cos(t)-1| \sqrt{(\cos t)^2 + (-\sin t)^2}\, dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} |\cos(t)-1| dt$$

$c)$

$$S.A = 2\pi \left[|\sin(t) - t| \right]_0^{\frac{\pi}{2}}$$

$$= 2\pi \left[\left|1 - \tfrac{\pi}{2}\right| - |0| \right] = 2\pi \left(\frac{\pi}{2} - 1 \right)$$

$Q17:$ Sketch the region R bounded by $y = \sqrt{1-x^2}$ and the line $y = 0$ and Find the volume of the solid that is formed by revolving the region R about *the* $x-axis$

solution:

Disk method

$$Volume = \int_{-1}^{1} \pi (radius)^2\, dx$$

$$= \int_{-1}^{1} \pi \left(\sqrt{1-x^2}\right)^2 dx$$

$$= \pi \int_{-1}^{1} (1-x^2) dx$$

$$= \pi \left[x - \frac{1}{3}x^3 \right]_{-1}^{1} = \frac{4\pi}{3}$$