King Saud University, College of Sciences, Department of Mathematics,
First Term 1435/1436, Final Math 106

Q. No: 1 [1 Marks] Evalute $\int (\cos(x) + 2)^5 \sin(x)dx$

Q. No: 2 [1 Marks] If a point P has a polar coordinate $(r, \theta) = (2, \frac{\pi}{4})$, find then its rectangular coordinate (x, y).

Q. No: 3 [1 Marks] Find the arc length of the curve $y = \cosh x$, where $0 \leq x \leq \ln 2$

Q. No: 4 [1 Marks] Evaluate the following improper integral $\int_0^\infty xe^{-3x^2}dx$

Q. No: 5 [1 Marks] Evalute $\int 2^{-2x^2}xdx$

Q. No: 6 [3 Marks] Let P be the partition of $[2, 6]$ determined by $\{2, 3, 4, 6\}$. Find a Riemann sum R_p for the given function $f(x) = 4 - 5x$ by choosing, in each subinterval of P : a)The right-hand endpoint.
b)The left-hand endpoint.

Q. No: 7 [3 Marks] **Evaluate** $\int \frac{dx}{\sqrt{-5 - 6x - x^2}}dx$

Q, No: 8 [5 Marks] **Evaluate** $\int \frac{3x^2 + 20x + 28}{x^2 + 6x + 8}dx$

Q. No: 9 [4 Marks] **Evaluate** $\int e^x \cos(x)dx$

Q No: 10 [5 Marks] **Sketch** the region R that is outside of the curve $r = 3$ and inside the curve $r = 2 + 2\cos\theta$ and find its area.

Q No: 11 [5 Marks] **Find** the area of the surface generated by revolving about the $x-$axis the curve $C : x = t,\ y = 2\sqrt{t}$ where $1 \leq t \leq 2$

Q. No: 12 [5 Marks] Let R be the region bounded by the graph $x = -y^2 + 4$, $y - x = 2$, $y = 1$. **Sketch** the region R and **find** the **volume** of the solid R generated by revolving the region R about the $x-$axis.

Q, No: 13 [5 Marks] **Sketch** the graph of the equation $r = 1 + \cos(\theta)$, and **find** the area of the surface of revolution generated by revolving the curve $r = 1 + \cos(\theta)$, where $\theta \in [0, \pi]$ about the polar axis.

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Q. No: 1 [1 Marks] Evalute $\int (\cos(x)+2)^5 \sin(x)dx$

Let $u = \cos(x) + 2$ then $du = -\sin(x)dx$, [0.5]

and we will have then

$$\begin{aligned}\int (\cos(x)+2)^5 \sin(x)dx &= -\int u^5 du = -\frac{u^6}{6} + c \\ &= -\frac{(\cos(x)+2)^6}{6} + c \text{ [0.5]}\end{aligned}$$

Q. No: 2 [1 Marks] If a point P has a polar coordinate $(r,\theta) = (2, \frac{\pi}{4})$, find then its rectangular coordinate (x, y).

$$\begin{aligned} x &= r\cos(\theta) = 2\cos\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \text{[0.5]} \\ y &= r\sin(\theta) = 2\sin\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \text{[0.5]}\end{aligned}$$

then $(x, y) = (\sqrt{2}, \sqrt{2})$.

Q. No: 3 [1 Marks] Find the arc length of the curve $y = \cosh x$, where $0 \le x \le \ln 2$

We have

$$\begin{aligned} L_0^{\ln 2} &= \int_0^{\ln 2} \sqrt{1+\sinh^2 x}dx \text{ [0.5]} \\ &= \frac{3}{4}.\text{[0.5]}\end{aligned}$$

Q. No: 4 [1 Marks] Evaluate the following improper integral $\int_0^{\infty} xe^{-3x^2}dx$

We have

$$\int_0^{\infty} xe^{-3x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-3x^2}dx \text{ [0.25]}$$

Let $u = -3x^2$, then $du = -6xdx$ [0.25] and we have $\int xe^{-3x^2}dx = \int \frac{-1}{6}e^u du = \frac{-1}{6}e^{-3x^2} + C,$ [0.25]

and hence we get

$$\lim_{t\to\infty}\int_0^t xe^{-3x^2}dx = \frac{1}{6}[0.25]$$

Q. No: 5 [1 Marks] Evalute $\int 2^{-2x^2}xdx$

Let $u = -2x^2$ then $-\frac{1}{4}du = xdx$, [0.5] and we have

$$\int 2^{-2x^2}xdx = -\frac{1}{4}\int 2^u du = -\frac{1}{4}\frac{2^u}{\ln 2} + c = -\frac{1}{4}\frac{2^{-2x^2}}{\ln 2} + c \quad [0.5]$$

Q. No: 6 [3 Marks] Let P be the partition of $[2,6]$ determined by $\{2,3,4,6\}$. Find a Riemann sum R_p for the given function $f(x) = 4 - 5x$ by choosing, in each subinterval of P :

We have 3 subintervals $[2,3]$, $[3,4]$, $[4,6]$, and

$x_0 = 2, x_1 = 3, x_3 = 4, x_4 = 6, \Delta x_1 = 1, \Delta x_2 = 1, \Delta x_3 = 2, w_k \in [x_{k-1}, x_k], k = 1,2,3$[0.5]

$$R_p = \sum_{k=1}^{3} f(w_k)\Delta x_k \text{ [0.5]}$$

$a)$ The right-hand endpoint.

$$\begin{aligned} R_p &= f(3)(1) + f(4)(1) + f(6)(2) \text{ [0.5]} \\ &= 4 - (5)3 + 4 - (5)4 + (4 - (5)6)2 \\ &= -79 \text{ [0.5]} \end{aligned}$$

$b)$ The left-hand endpoint.

$$\begin{aligned} R_p &= f(2)(1) + f(3)(1) + f(4)(2)\text{[0.5]} \\ &= 4 - (5)2 + 4 - (5)3 + (4 - (5)4)2 \\ &= -49 \text{ [0.5]} \end{aligned}$$

Q. No: 7 [3 Marks] **Evaluate** $\int \frac{dx}{\sqrt{-5-6x-x^2}}$

We have

$$\begin{aligned}
\int \frac{dx}{\sqrt{-5-6x-x^2}} &= \int \frac{dx}{\sqrt{4-(x+3)^2}} \text{ [1]} \\
&= \int \frac{dx}{\sqrt{4-u^2}}, \ [u = x+3]\text{[1]} \\
&= \sin^{-1}(\frac{u}{2}) + c = \sin^{-1}(\frac{x+3}{2}) + c \text{ [1]}
\end{aligned}$$

Q, No: 8 [5 Marks] **Evaluate** $\int \frac{3x^2+20x+28}{x^2+6x+8} dx$

We have

$$\begin{aligned}
\int \frac{3x^2+20x+28}{x^2+6x+8} dx &= \int 3dx + \int \frac{2x+4}{x^2+6x+8} dx \text{ [2]} \\
&= \int 3dx + \int \frac{2}{x+4} dx \text{ [1]} \\
&= 3x + 2\ln|x+4| + c \text{ [1+1]}
\end{aligned}$$

Q. No: 9 [4 Marks] **Evaluate** $\int e^x \cos(x) dx$

We have

$$\begin{aligned}
\int e^x \cos(x) dx &= e^x \sin(x) - \int e^x \sin(x) dx \text{ [1]} \\
&= e^x \sin(x) - [e^x(-\cos(x)) + \int e^x \cos(x) dx \text{ [1]} \\
&= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx \\
&= \frac{1}{2} e^x [\sin x + \cos x] \text{ [2]}
\end{aligned}$$

Q No: 10 [5 Marks] **Sketch** the region R that is outside of the curve $r = 3$ and inside the curve $r = 2 + 2\cos\theta$ and find its area.

Graph [1]

As $2+2\cos\theta = 3$ then we have $\theta = \pi/3$ and $\theta = -\pi/3$ [1]

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}((2+\cos\theta)^2 - 3^2)d\theta[1] \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(\cos^2\theta + 4\cos\theta - 5)d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(\frac{1+\cos 2\theta}{2} + 4\cos\theta - 5)d\theta[1] \\ &= \frac{17}{8}\sqrt{3} - \frac{3}{2}\pi \text{ [1]} \end{aligned}$$

Q No: 11 [5 Marks] **Find** the area of the surface generated by revolving about the $x-$axis the curve $C: x = t,\ y = 2\sqrt{t}$ where $1 \le t \le 2$

We have

$$\begin{aligned} S.A &= 2\pi\int_1^2 |y|\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt \text{ [2]} \\ &= 2\pi\int_1^2 2\sqrt{t}\sqrt{1+\frac{1}{t}}dt \text{ [1+1]} \\ &= 4\pi\int_1^2 \sqrt{t+1}dt = 8\pi(\sqrt{3} - \frac{2}{3}\sqrt{2}) \text{ [1]} \end{aligned}$$

Q. No: 12 [5 Marks] Let R be the region bounded by the graph $x = -y^2 + 4$, $y - x = 2$, $y = 1$

Sketch the region R and **find** the **volume** of the solid R generated by revolving the region R about the $x-$axis.

Graph [2 Mark]

We have for $1 \le y \le 2$

$$\begin{aligned} V &= 2\pi\int_1^2 y(-y^2 + 4 - y + 2)dy \text{ [2]} \\ &= \frac{35}{6}\pi \text{ [1]} \end{aligned}$$

or

$$\begin{aligned} V &= \pi\int_{-1}^{0}((2+x)^2 - 1^2)dx + \pi\int_0^3((\sqrt{4-x})^2 - 1^2)dx \text{ [1+1]} \\ &= \frac{35}{6}\pi \text{ [1]} \end{aligned}$$

Q, No: 13 [5 Marks] **Sketch** the graph of the equation $r = 1 + \cos(\theta)$, and **find** the area of the surface of revolution generated by revolving the curve $r = 1 + \cos(\theta)$, where $\theta \in [0, \pi]$ about the polar axis.

Graph [1 Mark]

We have

$$\begin{aligned} S.A &= 2\pi \int_0^{\pi} |r \sin\theta| \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2} d\theta \ [2] \\ &= 2\pi \int_0^{\pi} |r \sin\theta| \sqrt{\cos^2\theta + 2\cos\theta + 1 + (-\sin\theta)^2} d\theta \\ &= 2\sqrt{2}\pi \int_0^{\pi} (1+\cos\theta)^{\frac{3}{2}} \sin(\theta) d\theta \ [u = \cos\theta] \ [0.5] \end{aligned}$$

we have $\int (1+\cos\theta)^{\frac{3}{2}} \sin(\theta) d\theta = -\int (1+u)^{\frac{3}{2}} du = -\frac{2}{5}(u+1)^{\frac{5}{2}} + c$ [1]

and then

$$\begin{aligned} S.A &= 2\sqrt{2}\pi \int_0^{\pi} (1+\cos\theta)^{\frac{3}{2}} \sin(\theta) d\theta \\ &= \frac{32}{5}\pi \ [0.5] \end{aligned}$$