King Saud University
College of Sciences
Department of Mathematics

First Semester (1434/1435)

Final Exam, M-106

Programmable Calculators are Not Authorized

Name:	Number:
Name of Teacher:	Group No:

Max Marks: 40 Time: Three hours

The Exam paper contains 8 pages
(**10 Multiple choice questions and 7 Full questions**)

Multiple Choice (1-10)	
Question 11	
Question 12	
Question 13	
Question 14	
Question 15	
Question 16	
Question 17	
Total	

Multiple Choice

Q.No:	1	2	3	4	5	6	7	8	9	10
$\{a, b, c, d\}$	b	c	b	d	d	b	a	b	b	b

Q. No: 1 The surface area of the region obtained by revolving $x = \frac{1}{3}y^3$, $0 \le y \le 1$ about the y-axis is:

(a) $\frac{2\sqrt{2}\,\pi}{9}$ (b) $\frac{\pi}{9}(2\sqrt{2}-1)$ (c) $\frac{\pi}{27}(10\sqrt{10}-1)$ (d) $\frac{\pi}{3}(2\sqrt{2}-1)$.

Q. No: 2 The value of the number z that satisfies the conclusion of the Mean Value Theorem of the function $f(x) = \ln x$ on the interval $[1, e]$ is

(a) $z = e - 1$ (b) $z = e^{e-1}$ (c) $z = e^{\frac{1}{e-1}}$ (d) $z = \frac{e-1}{2}$.

Q. No: 3 The slope of the tangent line to the graph of $y = 3^x - 2^{-x}$ at $x = 1$ is equal to:

(a) $\ln 3 + 2\ln(2)$ (b) $3\ln 3 + \frac{1}{2}\ln(2)$ (c) $\ln 3 + \ln(2)$ (d) $3\ln 3 - \frac{1}{2}\ln(2)$.

Q. No: 4 If a point has $xy-$coordinates $(x, y) = (\sqrt{3}, -1)$ then one of its (r, θ) coordinates is:

(a) $(-2, \frac{\pi}{3})$ (b) $(-2, \frac{\pi}{6})$ (c) $(2, -\frac{\pi}{3})$ (d) $(2, -\frac{\pi}{6})$.

Q. No: 5 The derivative of the function $F(x) = \int_{\cos(x)}^{\sin(x)} \frac{dt}{t + \sqrt{1-t^2}}$ is equal to:

(a) $\frac{1}{\cos(x) + \sin(x)}$ (b) 0 (c) $\frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)}$ (d) 1.

Q. No: 6 The arc length of the curve of $y = \frac{2}{3}x^{\frac{3}{2}}$ from the point $(1, \frac{2}{3})$ to the point $(4, \frac{16}{3})$ is equal to:

(a) $\frac{5}{3}\sqrt{5} - \frac{4}{3}\sqrt{2}$ (b) $\frac{10}{3}\sqrt{5} - \frac{4}{3}\sqrt{2}$ (c) $\frac{10}{3}\sqrt{5} + \frac{4}{3}\sqrt{2}$ (d) $\frac{5}{3}\sqrt{5} + \frac{4}{3}\sqrt{2}$.

Q. No: 7 The improper integral $\int_e^{\infty} \frac{1}{x\ln(x)}dx$

(a) diverges (b) converges to 0 (c) converges to 1 (d) converges to e.

Q. No: 8 The area of the region that is inside of the curve $r = 2$ and outside of the curve $r = 1$ its equal to:

(a) 4π (b) 3π (c) 2π (d) π.

Q. No: 9 The area of the region bounded by the curve $r = \sin(\theta)$ is given by:

(a) $\int_0^{2\pi} \pi \sin^2(\theta) d\theta$ (b) $\int_0^{\pi} \frac{1}{2} \sin^2(\theta) d\theta$ (c) $\int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2(\theta) d\theta$ (d) $\int_0^{2\pi} \frac{1}{2} \sin^2(\theta) d\theta$.

Q. No: 10 The integral $\int \tan^{-1}(x) dx$ is equal to :

(a) $\tan^{-1}(x) - \frac{1}{2} \ln(1 + x^2) + c$ (b) $x \tan^{-1}(x) - \frac{1}{2} \ln(1 + x^2) + c$

(c) $\tan^{-1}(x) + \frac{1}{2} \ln(1 + x^2) + c$ (d) $x \tan^{-1}(x) + \frac{1}{2} \ln(1 + x^2) + c$.

Full Questions

Question No: 11 **Approximate** $\int_0^1 \frac{2^x}{1+x}dx$ by using **trapezoidal** rule with $n = 4$. [**3 Marks**]

We have $[a, b] = [0, 1]$ and $n = 4$, then $\Delta x = \frac{1-0}{4} = 0.25$. [**1**]

$$
\begin{aligned}
\int_0^1 \frac{2^x}{1+x}dx &\simeq \frac{1}{8}[f(0) + 2f(\frac{1}{4}) + 2f(\frac{1}{2}) + 2f(\frac{3}{4} + f(1)], [\mathbf{1}] \\
&\simeq \frac{1}{8}[1 + 2(\frac{2^{\frac{1}{4}}}{\frac{5}{4}}) + 2(\frac{2^{\frac{1}{2}}}{\frac{3}{2}}) + 2(\frac{2^{\frac{3}{4}}}{\frac{7}{4}}) + \frac{2}{2}], [\mathbf{0.5}] \\
&\simeq \frac{1}{8}[7.7104] = 0.9638.[\mathbf{0.5}]
\end{aligned}
$$

Question No: 12 Evaluate $\int \sin^3(x)\cos^3(x)dx$. [**4 Marks**]

$$
\begin{aligned}
\int \sin^3(x)\cos^3(x)dx &= \int \sin^3(x)\cos^2(x)\cos(x)dx \\
&= \int \sin^3(x)(1 - \sin^2(x))\cos(x)dx.[\mathbf{1}]
\end{aligned}
$$

Put $u = \sin(x)$, then $du = \cos(x)dx$, [**1**] and

$$
\begin{aligned}
\int \sin^3(x)\cos^3(x)dx &= \int u^3(1 - u^2)du \\
&= \int (u^3 - u^5)du, [\mathbf{1}] \\
&= \frac{u^4}{4} - \frac{u^6}{6}, [\mathbf{0.5}] \\
&= \frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6} + c.[\mathbf{0.5}]
\end{aligned}
$$

Question No: 13 Evaluate $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$. [**4 Marks**]

$$
\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = \infty - \infty = "I.F", [\mathbf{1}]
$$

Then

$$
\begin{aligned}
\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) &= \lim_{x\to 0^+} \frac{\sin(x) - x}{x\sin(x)} = "\frac{0}{0}", [\mathbf{1}] \\
(By\, L'H.R) &= \lim_{x\to 0^+} \frac{\cos(x) - 1}{\sin(x) + x\cos(x)} = "\frac{0}{0}", [\mathbf{1}] \\
(By\, L'H.R) &= \lim_{x\to 0^+} \frac{\sin(x)}{2\cos(x) - x\sin(x)} = 0, [\mathbf{1}]
\end{aligned}
$$

Question No: 14 Evaluate the integral $\int (\frac{2x^2 - x + 4}{x^3 + 4x}) dx$. [**5 Marks**]

We have $\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$ with $A = 1$, $B = 1$ and $C = -1$.[**2 Marks**]

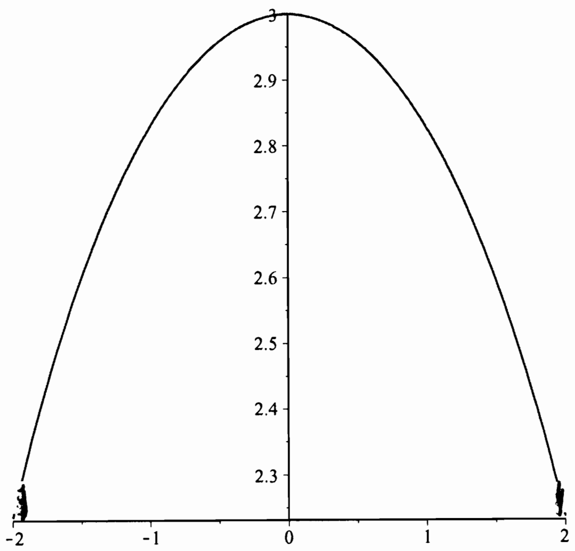
Then

$$\begin{aligned}
\int \frac{2x^2 - x + 4}{x^3 + 4x} dx &= \int \frac{1}{x} + \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4} dx \\
&= \int \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2 + 4} - \frac{1}{x^2 + 2^2} dx \\
&= \ln |x| + \frac{\ln(x^2 + 4)}{2} - \frac{\tan^{-1}(\frac{x}{2})}{2} + c.
\end{aligned}$$

[**1+1+1 Marks**]

Question No: 15 **Sketch** and **Find** the area of the surface generated by revolving about the x−axis the curve of $x = t, y = \sqrt{9 - t^2}$; $-2 \le t \le 2$. [**4 Marks**]

Graph [**1**]



$$
\begin{aligned}
L_{-2}^{2} &= \int_{-2}^{2} 2\pi|y|\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt, [\mathbf{1}] \\
&= \int_{-2}^{2} 2\pi\sqrt{9 - t^2}\sqrt{1 + \frac{t^2}{9 - t^2}}dt, [\mathbf{1}] \\
&= \int_{-2}^{2} 6\pi dt, [\mathbf{0.5}] \\
&= 24\pi. [\mathbf{0.5}]
\end{aligned}
$$

Question No: 16 Evaluate $\int (x - 3)\sqrt{-8 + 6x - x^2}\, dx$. [**5 Marks**]

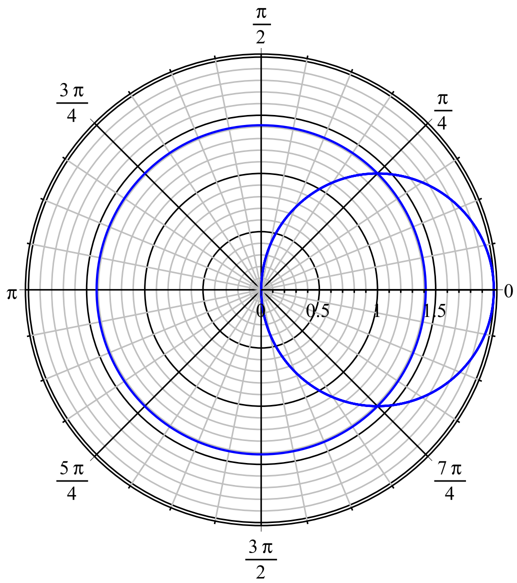
We have:

$$
\begin{aligned}
\int (x - 3)\sqrt{-8 + 6x - x^2}\, dx &= \int (x - 3)\sqrt{1 - (x - 3)^2}dx[\mathbf{1}] \\
&= \int u\sqrt{1 - u^2}du, \ (u = x - 3 \Rightarrow du = dx)[\mathbf{1}]
\end{aligned}
$$

Put $u = \sin(\theta)$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $du = \cos(\theta)d\theta$ and $\sqrt{1-u^2} = \cos(\theta)$[**1**].

$$\begin{aligned}
\int (x-3)\sqrt{-8+6x-x^2}\,dx &= \int \sin(\theta)\cos^2(\theta)d\theta, \\
&= -\frac{1}{3}\cos^3(\theta) + c, [\mathbf{1}] \\
&= -\frac{1}{3}(1-u^2)^{\frac{3}{2}} + c, [\mathbf{0.5}] \\
&= -\frac{1}{3}(1-(x-3)^2)^{\frac{3}{2}} + c[\mathbf{0.5}].
\end{aligned}$$

Question No: 17 **Sketch** and **find** the area of the region R that is inside of the curve $r = 2\cos(\theta)$ and the outside of the curve $r = \sqrt{2}$. [**5 Marks**].



Graph [**1**]
We have

$$\sqrt{2} = 2\cos(\theta) \Rightarrow \cos(\theta) = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}, \text{ or } \theta = -\frac{\pi}{4}[\mathbf{1}]$$

The area of the region R:

$$\begin{aligned} R &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\cos(\theta))^2 - (\sqrt{2})^2 d\theta[\mathbf{1}] \\ &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\cos(2\theta) + 2 - 2) d\theta[\mathbf{1}] \\ &= \frac{1}{2}\Big[\sin(2\theta)\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 1.[\mathbf{1}] \end{aligned}$$

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