Chapter 4 concepts

☺ **Independence of directions**

(relation of 2D to 1D motion)

☺ **Projectile motion**

☺ **Uniform circular motion**

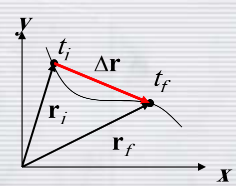
4. Motion in Two-Dimensions

4.1 The Displacement, Velocity, and Acceleration Vectors

Displacement in a plane

The displacement vector **r**:

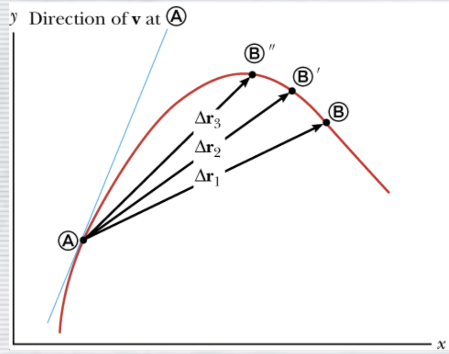
$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i$$



Displacement is the straight line between the final and initial position of the particle.

That is the vector difference between the final and initial position.

Average Velocity

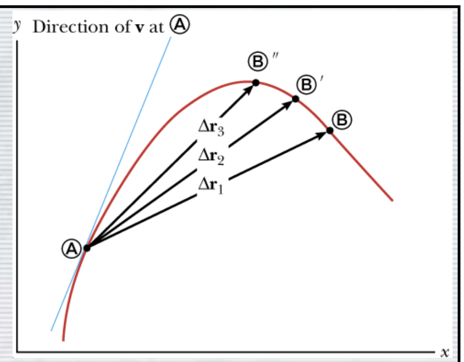


Average velocity **v**:

$$\Delta\bar{\vec{v}} \equiv \frac{\Delta \vec{r}}{\Delta t}$$

Average velocity: Displacement of a particle, Δ**r**, divided by time interval Δt.

Instantaneous Velocity



Instantaneous velocity **v**:

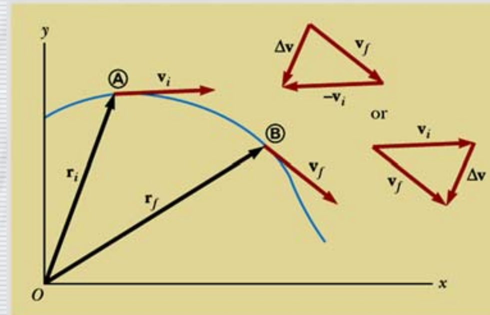
$$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Instantaneous velocity **v**: Limit of the average velocity as Δt approaches zero.

The instantaneous velocity equals the derivative of the position vector with respect to time.

The magnitude of the instantaneous velocity vector $v \equiv |\vec{v}|$ is called the **speed** (scalar)

Average Acceleration



Average acceleration:

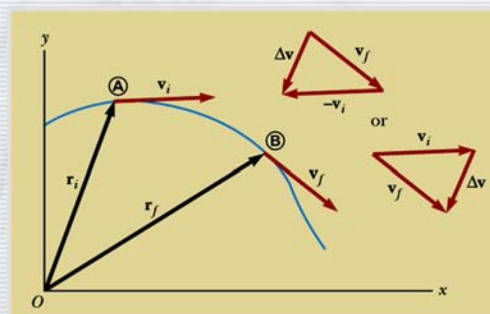
$$\bar{\vec{a}} \equiv \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\Delta\vec{v}}{\Delta t}$$

Average acceleration: Change in the velocity Δ**v** divided by the time Δt during which the change occurred.

Change can occur in direction and magnitude!

Acceleration points along **change in velocity Δv!**

Instantaneous Acceleration



Instantaneous acceleration:

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Instantaneous acceleration: limiting value of the ratio $\frac{\Delta\vec{v}}{\Delta t}$ as Δt goes to zero.

Instantaneous acceleration equals the derivative of the velocity vector with respect to time.

4.2 Two-dimensional motion with **constant** acceleration **a**

Trick 1:

The equations of motion (kinematic equations) we derived before are still valid, but are now in vector form.

Trick 2 (Superposition principle):

Vector equations can be broken down into their x- and y-components. Then calculated independently.

Position vector:

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Velocity vector:

$$\vec{v} = v_x\vec{i} + v_y\vec{j}$$

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

Dr. Abdallah M.Azzeer 7

Assume a constant acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt} = \langle \vec{a} \rangle \quad \text{: constant}$$

$$\frac{\vec{v} - \vec{v}_0}{t - 0} = \vec{a} \implies \boxed{\vec{v} = \vec{v}_0 + \vec{a}t}$$

$$\vec{v} = \frac{d\vec{r}}{dt} \quad , \quad d\vec{r} = \vec{v} \cdot dt$$

$$\int d\vec{r} = \int_0^t \vec{v} \cdot dt$$

$$\vec{v} = \frac{d\vec{r}}{dt} \quad , \quad d\vec{r} = \vec{v} \cdot dt$$

$$\int d\vec{r} = \int_0^t \vec{v} \cdot dt$$

$$\vec{r} - \vec{r}_0 = \int_0^t (\vec{v}_0 + \vec{a}t) \cdot dt = \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2$$

$$\boxed{\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2}$$

Dr. Abdallah M.Azzeer 8

In (x,y)-coordinates

$$\left.\begin{aligned}\vec{a} &= a_x\hat{i} + a_y\hat{j},\\ \vec{v} &= v_x\hat{i} + v_y\hat{j},\\ \vec{v}_0 &= v_{0x}\hat{i} + v_{0y}\hat{j},\\ \vec{r} &= x\hat{i} + y\hat{j},\\ \vec{r}_0 &= x_0\hat{i} + y_0\hat{j}\end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned}v_x &= v_{0x} + a_x t\\ v_y &= v_{0y} + a_y t\end{aligned}}$$

$$\boxed{\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2}$$

$$x\hat{i} + y\hat{j} = x_0\hat{i} + y_0\hat{j} + v_{0x}t\hat{i} + v_{0y}t\hat{j} + \tfrac{1}{2}a_x t^2\hat{i} + \tfrac{1}{2}a_y t^2\hat{j}$$

$$x\hat{i} + y\hat{j} = (x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2)\hat{i} + (y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2)\hat{j}$$

$$\Longrightarrow \boxed{\begin{aligned}x &= x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2\\ y &= y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2\end{aligned}}$$

Read example 4.1

Dr. Abdallah M.Azzeer 9

Example

A particle starts at origin when t=0 with an initial velocity $\vec{v} = (20\hat{i} - 15\hat{j})$ *m/s*. The particle moves in the xy plane with $a_x = 4.0\ m/s^2$.

(a) Determine the components of velocity vector at any time, t.

(b) Compute the velocity and speed of the particle at t=5.0 s

(c) Determine the x and y components of the particle at t=5.0 s.

(d) Can you write down the position vector at t=5.0s?

(a)

$$\boxed{v_{xf} = v_{xi} + a_x t = 20 + 4.0\,t\ (m/s)} \quad \boxed{v_{yf} = v_{yi} + a_y t = -15\ (m/s)}$$

$$\boxed{\vec{v}(t) = \{(20 + 4.0\,t)\hat{i} - 15\hat{j}\}\ m/s}$$

Dr. Abdallah M.Azzeer 10

(b)

$$\boxed{\vec{v} = \{(20 + 4.0\times5.0)\hat{i} - 15\hat{j}\}\ m/s = (40\hat{i} - 15\hat{j})\ m/s}$$

$$speed = |\vec{v}| = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(40)^2 + (-15)^2} = 43m/s$$

Magnitude

Direction

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-15}{40}\right) = \tan^{-1}\left(\frac{-3}{8}\right) = -21^\circ$$

(c)

$$x_f = v_{xi}t + \frac{1}{2}a_x t^2 = 20\times5 + \frac{1}{2}\times4\times5^2 = 150(m)$$

$$y_f = v_{yi}t = -15\times5 = -75\ (m)$$

(d)

$$\vec{r}_f = x_f\hat{i} + y_f\hat{j} = (150\hat{i} - 75\hat{j})m$$

Now in a real Problem

Projectile Motion

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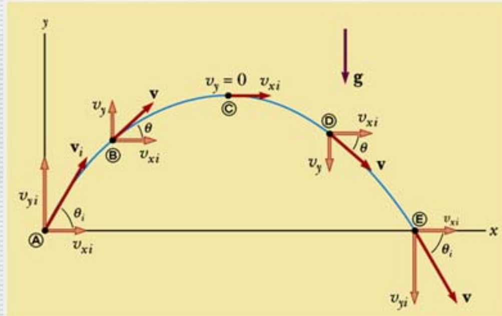
4.3 Projectile Motion

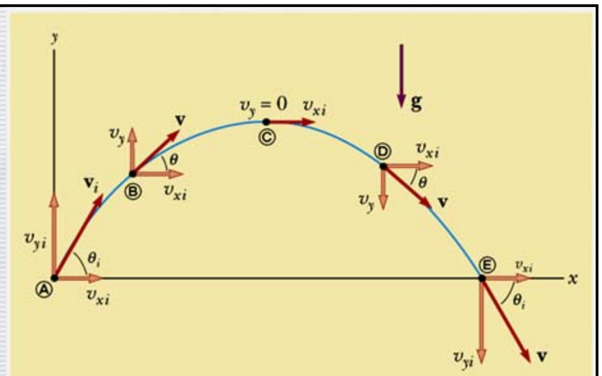
Two assumptions:

1. Free-fall acceleration **g** is constant.
2. Air resistance is negligible.



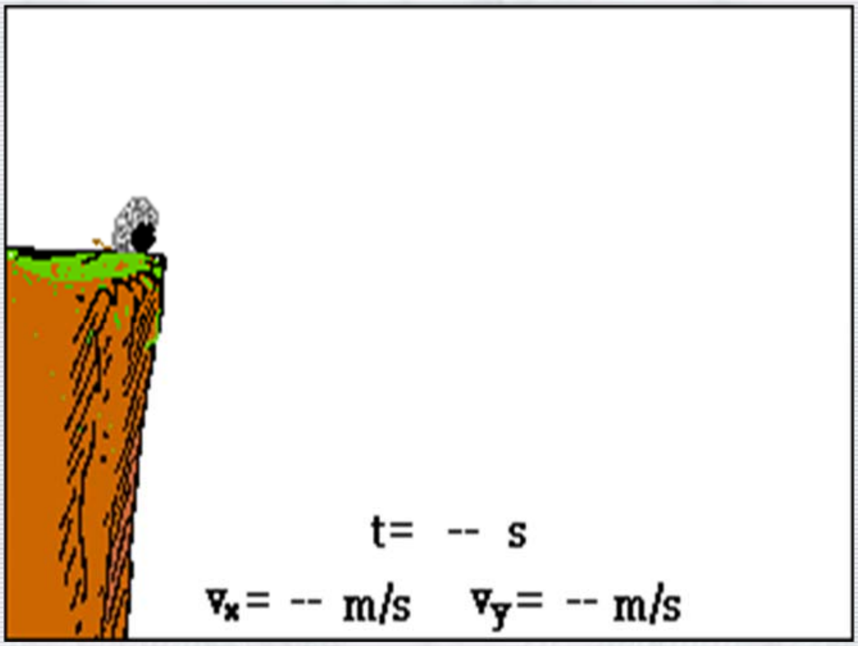
- The path of a projectile is a parabola (derivation: see book).
- Projectile leaves origin with an initial velocity of $\mathbf{v}_i$.
- Projectile is launched at an angle θ_i
- Velocity vector changes in magnitude and direction.
- Acceleration in y-direction is **g**.
- Acceleration in x-direction is 0.

Projectile motion

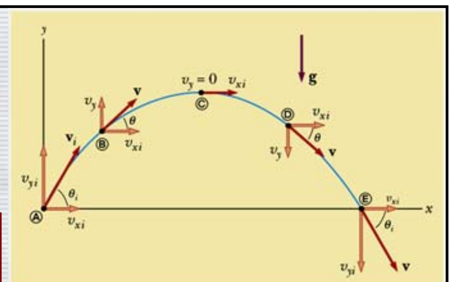


Superposition of motion in x-direction and motion in y-direction

Acceleration in x-direction is 0. (Constant velocity)	Acceleration in y-direction is **g**. (Constant acceleration)
$x_f = x_i + v_{xi} t$	$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$
$v_{xf} = v_{xi}$	$v_{yf} = v_{yi} + gt$



Equations



- *x- Component*

$$x_f = x_i + v_{xi} t$$

- *y- Component*

$$y_f = y_i + v_{yi} t - \frac{1}{2} g t^2$$

$$v_{yf}^{\ 2} = v_{yi}^{\ 2} - 2g\Delta y$$

- *Vectors*

$$v_{yf} = v_{yi} - gt$$

$$v_{xi} = v_i \cos(\theta)$$

$$v_{yi} = v_i \sin(\theta)$$

Vertical: $a_y = -g$

$$\frac{v_{yi} + v_{yf}}{2} = \frac{\Delta y}{t}$$
$$v_{yf} = v_{yi} - gt$$
$$\Delta y = v_{yi} t - \tfrac{1}{2} g t^2$$
$$v_{yf}^2 = v_{yi}^2 - 2g\Delta y$$

Horizontal: $a_x = 0$

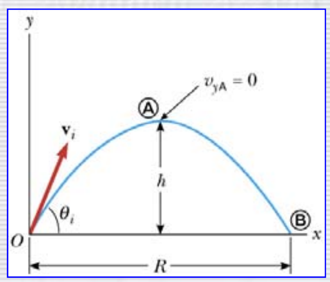
$$v_{xf} = v_{xi} = v_x$$
$$\Delta x = v_{xi}\, t$$

The two components are independent, but linked by the common time, *t*.

Horizontal Range and Max Height

Based on what we have learned in the previous pages, one can analyze a projectile motion in more detail

- Maximum height an object can reach
- Maximum range



At the maximum height the object's vertical motion stops to turn around!!

$$v_{yf} = v_{yi} + a_y t$$
$$= v_i \sin\theta_i - g t_A = 0$$

$$\therefore t_A = \frac{v_i \sin\theta_i}{g}$$

$$y_f = h = v_{yi}t + \frac{1}{2}(-g)t^2$$

$$= v_i \sin\theta_i \left(\frac{v_i \sin\theta_i}{g}\right) - \frac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$\boxed{y_f = \left(\frac{v_i^2 \sin^2\theta_i}{2g}\right)}$$

Since no acceleration in x, it still flies even if $v_y=0$

$$R = v_{xi}(2t_A) = 2v_i \cos\theta_i \left(\frac{v_i \sin\theta_i}{g}\right)$$

$$\boxed{R = \left(\frac{v_i^2 \sin 2\theta_i}{g}\right)}$$

Maximum Range and Height

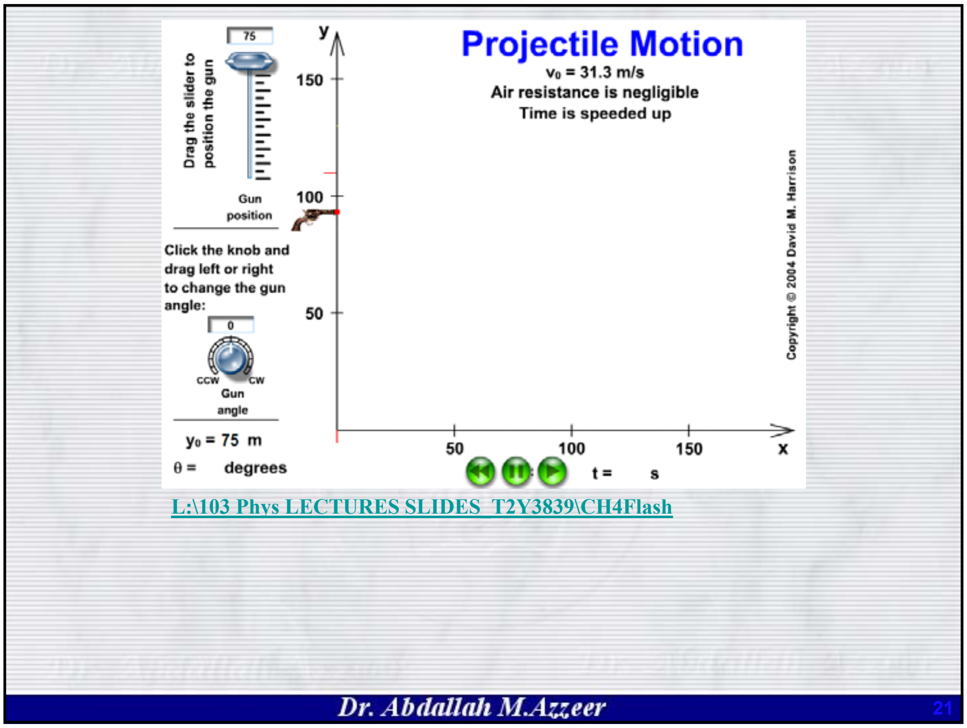
- What are the conditions that give maximum height and range of a projectile motion?

$$\boxed{h = \left(\frac{v_i^2 \sin^2\theta_i}{2g}\right)}$$

This formula tells us that the maximum height can be achieved when θ_i=90°!!!

$$\boxed{R = \left(\frac{v_i^2 \sin 2\theta_i}{g}\right)}$$

This formula tells us that the maximum range can be achieved when $2\theta_i$=90°, i.e., θ_i=45°!!!


Projectile Motion
v0 = 31.3 m/s
Air resistance is negligible
Time is speeded up
Drag the slider to position the gun
75
Gun position
Click the knob and drag left or right to change the gun angle:
0
CCW
CW
Gun angle
y0 = 75 m
θ = degrees
t = s
y
x
150
100
50
50
100
150
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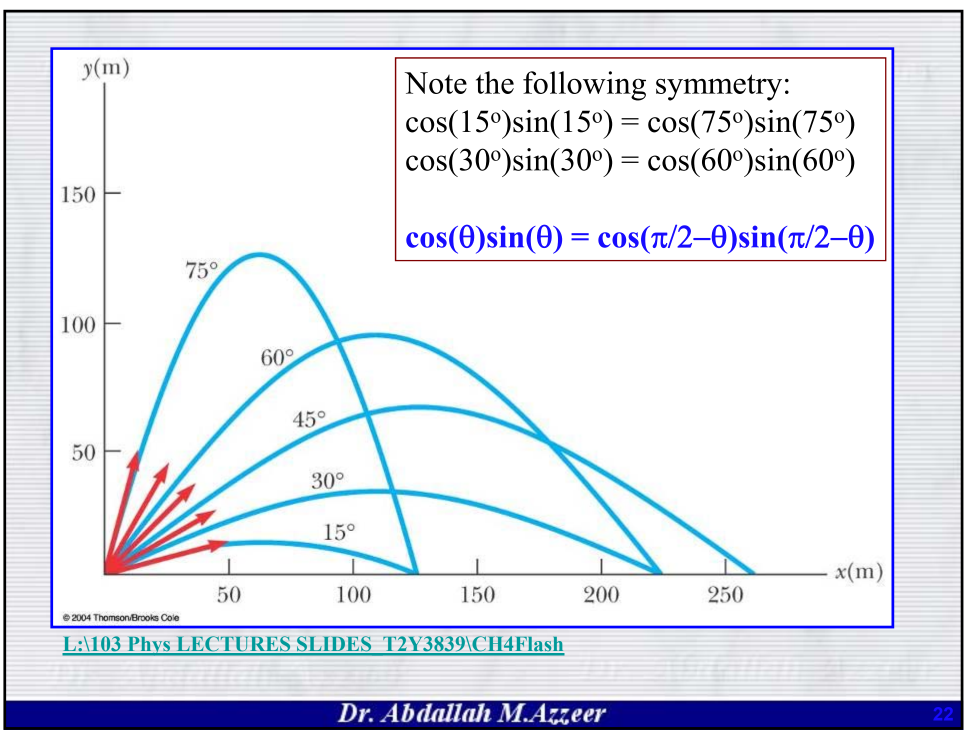
21


Note the following symmetry:
cos(15º)sin(15º) = cos(75º)sin(75º)
cos(30º)sin(30º) = cos(60º)sin(60º)
cos(θ)sin(θ) = cos(π/2–θ)sin(π/2–θ)
y(m)
x(m)
150
100
50
50
100
150
200
250
75°
60°
45°
30°
15°
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22

Solving Problems Involving Projectile Motion

- Read the problem carefully, and choose the object(s) you are going to analyze.
- Draw a diagram.
- Choose an origin and a coordinate system.
- Decide on the time interval; this is the same in both directions, and includes only the time the object is moving with constant acceleration *g*.
- Examine the *x* and *y* motions separately.
- List known and unknown quantities. Remember that v_x never changes, and that $v_y = 0$ at the highest point.
- Plan how you will proceed. Use the appropriate equations; you may have to combine some of them.