1.7 Diagonal, Triangular, and Symmetric Matrices

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is called a **diagonal matrix**.

some examples :

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A general n × n diagonal matrix **D** can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is **invertible** if and only if all of its diagonal entries are nonzero

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Powers of diagonal matrices are easy to compute

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

EXAMPLE 1:

Find $A^{-1}, A^{5}, and\ A^{-5}, If\ \ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Triangular Matrices

1) A square matrix in which all the entries above the main diagonal are zero is called **lower triangular**,
2) A square matrix in which all the entries below the main diagonal are zero is called **upper triangular**.
3) A matrix that is either upper triangular or lower triangular is called **triangular.**

EXAMPLE 2

Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 lower triangular matrix

Remark

Observe that diagonal matrices are both upper triangular and lower triangular since they have zeros below and above the main diagonal. Observe also that a square matrix in row echelon form is upper triangular since it has zeros below the main diagonal.

Properties of Triangular Matrices

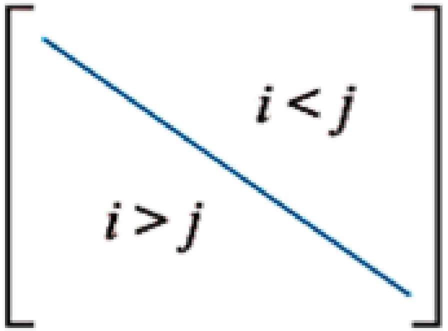
Example2illustratesthefollowingfourfactsabouttriangularmatricesthatwewillstate without formal proof:

- A square matrix $A = [\, a_{ij}]$is **upper triangular** if and only if all entries to the left of the main diagonal are zero; that is, a_{ij} =0 if I > j (Figure 1.7.1).

- A square matrix $A = [\, a_{ij}]$ is **lower triangular** if and only if all entries to the right of the main diagonal are zero; that is, a_{ij} =0 if I < j (Figure 1.7.1).

- A square matrix $A = [\, a_{ij}]$ is **upper triangular** if and only if the ith row starts with at least i−1 zeros for every i.

- A square matrix $A = [\, a_{ij}]$ is **lower triangular** if and only if the jth column starts with at least j −1 zeros for every j.



▲ Figure 1.7.1

The following theorem lists some of the basic properties of triangular matrices.

THEOREM:

(a) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

(b) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

(d) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

EXAMPLE 3

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows from part (c) of Theorem 1.7.1 that the matrix **A** is invertible but the matrix **B** is not. Moreover, the theorem also tells us that $\boldsymbol{A^{-1}}$, **AB**, and **BA** must be upper triangular. We leave it for you to confirm these three statements by showing that

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

Symmetric Matrices

Definition:
A square matrix A is said to be **symmetric** if $A = A^T$

EXAMPLE 4
The following matrices are symmetric, since each is equal to its own transpose.

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

THEOREM:

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

(a) AT is symmetric.

(b) A+B and A−B are symmetric.

(c) kA is symmetric.

THEOREM:

The product of two symmetric matrices is symmetric if and only if the matrices commute.

Invertibility of Symmetric Matrices

THEOREM:

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Products $A\,A^T$ and $A^T\,A$ are Symmetric

EXAMPLE 6

The Product of a Matrix and Its Transpose Is Symmetric

Let A be the 2 ×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Solution:

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that $\boldsymbol{A\,A^T}$ and $\boldsymbol{A^T\,A}$ are symmetric as expected

THEOREM:

If **A** is an invertible matrix, then $\boldsymbol{A}\,\boldsymbol{A}^{\boldsymbol{T}}$ and $\boldsymbol{A}^{\boldsymbol{T}}\,\boldsymbol{A}$ are also invertible

Exercise Set 1.7

► In Exercises 1–2, classify the matrix as upper triangular, lower triangular, or diagonal, and decide by inspection whether the matrix is invertible. [*Note:* Recall that a diagonal matrix is both upper and lower triangular, so there may be more than one answer in some parts.] ◄

1. (a) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -2 & 7 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix}$

2. (a) $\begin{bmatrix} 4 & 0 \\ 1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 0 & 0 \end{bmatrix}$

► In Exercises 3–6, find the product by inspection. ◄

3. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -5 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 & 4 & -4 \\ 1 & -5 & 3 & 0 & 3 \\ -6 & 2 & 2 & 2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 1 & 2 & 0 \\ -5 & 1 & -2 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

► In Exercises 7–10, find A^2, A^{-2}, and A^{-k} (where k is any integer) by inspection. ◄

7. $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

8. $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

9. $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

10. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$