ON THE SPECTRUM OF THE TENSOR PRODUCTS OF JC-ALGEBRAS

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Abstract. We study in this article the relationship between the spectrum $\sigma(A)$ of a JC-algebra A and the spectrum $\sigma(C^*(A))$ of its universal enveloping C*-algebra $C^*(A)$ (see below for definition). Also we establish the Jordan analogue of some results in the context of spectra of tensor products C*-algebras. In particular, we extend Guichardet's results [**5**, Proposition 5] to JC-algebras.

1. Introduction

The definition of a tensor product of JC-algebras was first introduced by Hanche-Olsen [**7**] and then a progress in the theory was made by Jamjoom [**9**, **10**] who introduced a more general definition which provided a convenient treatment of tensor products of JC-algebras and stressed the important connection between the theory of tensor products of JC-algebras and tensor products of C*-algebras. As the relation between JC-algebras and their universal enveloping C*-algebras is very strong (see [**2**, **3**, **8**, **9**, **11**]), we rely on the theory of tensor products of C*-algebras throughout this paper and we assume familiarity with the general theory as contained in [**5**, **12**, **17**]. We deal with C*-algebras having an identity. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C*-algebras, their minimal and maximal C*-tensor products are denoted by $\mathfrak{A}_1 \underset{\min}{\otimes} \mathfrak{A}_2$ and $\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2$, respectively. Given a *-representation $\{\pi_i, H_i\}$ of a C*-algebra $\mathfrak{A}_i, i = 1, 2$, let $\pi_1 \underset{\lambda}{\otimes} \pi_2$ denote the unique *-representation $\pi_1 \underset{\lambda}{\otimes} \pi_2 : \mathfrak{A}_1 \underset{\lambda}{\otimes} \mathfrak{A}_2 \to B(H_1 \otimes H_2)$ of $\mathfrak{A}_1 \underset{\lambda}{\otimes} \mathfrak{A}_2$ on $H_1 \otimes H_2$ defined by $(\pi_1 \otimes \pi_2)(x_1 \otimes x_2) = \pi_1(x_1) \otimes \pi_2(x_2)$, $x_i \in \mathfrak{A}_i$, where λ is the minimum or the maximum C*-norm [**17**, 4.4.7, 4.4.22].

The state space of a C*-algebra $\mathfrak{A}$ will be denoted by $S(\mathfrak{A})$. It is convex and weak* compact [**11**, p. 257]. The extreme points of $S(\mathfrak{A})$ are called **pure states,** and the set of all pure states will be denoted by $P(\mathfrak{A})$. If $\rho \in S(\mathfrak{A})$, then π_ρ will denote the representation of $\mathfrak{A}$ on the Hilbert space H_ρ, with cyclic vector ξ_ρ, obtained by the GNS construction (see [**11**, 4.5.2]).

A JC-algebra A is a norm (uniformly) closed Jordan subalgebra of the Jordan algebra $B(H)_{s.a}$ of all bounded self adjoint operators on a Hilbert space H. The Jordan product is given by $a \circ b = (ab + ba)/2$, where juxtaposition denotes the ordinary operator multiplication. The centre of a JC-algebra is the set $Z(A) = \{a \in A : ab = ba \ \forall b \in A\}$ [**18**, Proposition 1]. A subspace I of a JC-algebra A is

1991 *Mathematics Subject Classification.* Primary 46L10, 46L05, 47D25; Secondary 81R10.
Key words and phrases. C*−algebras, JC-algebras, Jordan algebras, tensor products of operator algebras.

called **a Jordan ideal** if $a \circ b \in I$ for every $a \in A$ and every $b \in I$. **A JW-algebra** $M \subseteq B(H)_{s.a}$ is a weakly closed JC-algebra. **The central support** of an element a in a JW-algebra M is the smallest central projection e in M such that $ea = a$ [**18**, p. 11]. A projection e of M is said to be **abelian** if eMe is associative, and it is called **minimal** if it is non-zero and contains no other non-zero projection of M, or equivalently, e is minimal if and only if $eMe = \mathbb{R}e$. **A JW-factor** is a JW-algebra with trivial centre; **a type I JW-factor** is a JW-factor which contains a minimal projection. A JW-algebra is said to be of **type** I_n if there is a family of abelian projections $(e_\alpha)_{\alpha \in J}$ such that $c_M(e_\alpha) = 1_M$, $\sum\limits_{\alpha \in J} e_\alpha = 1_M$ and card $J = n$. It is known that (see [**8**, Section 5.3]) each type I JW-factor is of type I_n, for some n, probably infinite. **A spin factor** V_k is a $(k+1)$-dimensional JW-factor of type I_2.

A linear map $\varphi : A \to B$ between JC-algebras A and B is called **a (Jordan) homomorphism** if it preserves the Jordan product. A Jordan homomorphism which is one to one is called **a Jordan isomorphism**. **A (concrete) representation** of a JC-algebra A is a (Jordan) homomorphism $\pi : A \to B(H)_{s.a}$, for some complex Hilbert space H. It is said to be **dense** if $\pi(A)^- = B(H)_{s.a}$. **A factor representation** of a JC-algebra A is a (Jordan) homomorphism of A onto a weakly dense subalgebra of a JW-factor M. **Type I factor representations** are defined accordingly. Two factor representations $\phi_i : A \to M_i$, where M_i is a type I JW-factor, $i = 1, 2$, are said to be **Jordan equivalent** if there is an isomorphism Ψ of M_1 onto M_2 such that $\phi_2 = \Psi\phi_1$. Each type I factor representation of a JC-algebra A is Jordan equivalent to $\phi_\rho : A \to c(\rho)A^{**}(a \mapsto c(\rho)a)$, where A^{**} is the second dual of A, ρ is a pure state of A and $c(\rho)$ the central support of ρ being a minimal central projection in A^{**} [**1**, Proposition 2.2], [**8**, 4.5.7, 4.6.2, 4.6.4].

A JC-algebra A is said to be **reversible** if $a_1 a_2 ... a_n + a_n a_{n-1} ... a_1 \in A$ whenever $a_1, a_2, ..., a_n \in A$, and is said to be **universally reversible** if $\pi(A)$ is reversible for every representation π of A [**7**, p. 5]. A spin factor V is universally reversible when $\dim V = 3$ or 4, non-reversible when $\dim V \neq 3, 4$ or 6, and it can be either reversible or non-reversible if $\dim V = 6$ [**1**, p. 280], [**8**, 6.2.5]. Every JW-algebra without a direct summand of type I_2 is universally reversible [**8**, 5.1.5, 5.3.5, 6.2.3], and a JC-algebra is universally reversible if and only if it has no type I factor representation onto a spin factor other than V_2 and V_3 [**7**, Theorem 2.2].

If A is a JC-algebra, let $C^*(A)$ be **the universal enveloping C*-algebra of** A, and let Φ_A be **the canonical involutory *-antiautomorphism of** $C^*(A)$. Usually we will regard A as a generating Jordan subalgebra of $C^*(A)$ so that Φ_A fixes each point of A(see [**8**, Theorem 7.1.8]) for the existance and properties of $C^*(A)$). It is known [**7**, Lemma 4.2] that a JC-algebra A is universally reversible if and only if it is reversible in $C^*(A)$.

The reader is referred to [**3, 7, 8, 14, 15**] for a detailed account of the theory of JC-algebras and JW-algebras. The relevant background on the theory of C*-algebras and von Neumann algebras can be found in [**4**, **11**, **12**, **17**].

Definition 1. *Let A and B be a pair of JC-algebras canonically embedded in their respective universal enveloping* C^**-algebras* $C^*(A)$ *and* $C^*(B)$ *and let* λ *be a* C^**-norm on* $C^*(A) \otimes C^*(B)$*. Then* ***the JC-tensor product of*** A ***and*** B ***with***

respect to λ *is the completion* $JC(A \underset{\lambda}{\otimes} B)$ *of the real Jordan algebra* $J(A \otimes B)$ *generated by* $A \otimes B$ *in* $C^*(A) \underset{\lambda}{\otimes} C^*(B)$.

Theorem 1 (**9**, Corollary 2.3)**.** *Let* A *and* B *be JC-algebras. Then*

$$C^*(JC(A \underset{\lambda}{\otimes} B)) = C^*(A) \underset{\lambda}{\otimes} C^*(B)$$

where λ *is the minimum (min) or the maximum (max)* C^**-norm.*

Theorem 2 (**9**, Proposition 4.1)**.** *Let* A_i *be a JC-algebra and* $\pi_i : A_i \to B(H_i)$ *be a Jordan homomorphism, where* H_i *is a complex Hilbert space. Then there is a unique Jordan homomorphism of* $JC(A_1 \underset{\min}{\otimes} A_2)$ *into* $B(H_1 \otimes H_2)$ *such that* $(\pi_1 \underset{\min}{\otimes} \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$, $a_i \in A_i$, $i = 1, 2$. *Furthermore, if* π_i *is injective and* A_i *has no representations onto spin factors of the form* V_{4n+1}, $n < \infty$, *then* $\pi_1 \underset{\min}{\otimes} \pi_2$ *is injective.*

Theorem 3 (9, Proposition 1.2)**.** *Let* A_i *and* C *be JC-algebras and let* $\pi_1 : A_1 \to C$, $\pi_2 : A_2 \to C$ *be pointwise operator commuting Jordan homomorphisms,* $i = 1, 2$. *Then there exists a unique Jordan homomorphism* $\pi_1 \underset{\max}{\otimes} \pi_2 : JC(A_1 \underset{\max}{\otimes} A_2) \to C$ *such that* $(\pi_1 \underset{\max}{\otimes} \pi_2)(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2) = \pi_2(a_2)\pi_1(a_1) = \pi_1(a_1) \circ \pi_2(a_2)$, $a_i \in A_i$.

The reader is referred to [**9**] for the properties of the JC-tensor product of JC-algebras.

2. The Spectrum

Recall that two representations $\{\pi_1, H_1\}$ and $\{\pi_2, H_2\}$ of a C*-algebra $\mathfrak{A}$ are said to be **spatially equivalent** (or **unitarily equivalent**) if there is an isometry u of H_1 onto H_2 such that $u\pi_1(x)u^* = \pi_2(x), x \in \mathfrak{A}$. They are **equivalent** if there is an isomorphism ψ of $\pi_1(\mathfrak{A})^-$ onto $\pi_2(\mathfrak{A})^-$ such that $\psi(\pi_1(x)) = \pi_2(x), x \in \mathfrak{A}$, where $\pi_i(\mathfrak{A})^-$ is the strong (= weak) closure of $\pi_i(\mathfrak{A})$ in $B(H_i)$, $i = 1, 2$. It is clear that spatially equivalent representations are equivalent [**13**, 3.3.6]). A representation $\{\pi, H\}$ of a C*-algebra $\mathfrak{A}$ is called **irreducible** if the only subspaces of H invariant under π are H and $\{0\}$, or equivalently, $\pi(\mathfrak{A})^- = B(H)$ [**13**, 3.13.2]. Thus, every irreducible representation of a C*-algebra $\mathfrak{A}$ is a type I factor representation. Two irreducible representations of a C*-algebra $\mathfrak{A}$ are either disjoint or spatially equivalent [**13**, 3.13.3]. The cyclic representation $\{\pi_\omega, H_\omega, \xi_\omega\}$ associated with a pure state ω of a C*-algebra $\mathfrak{A}$ by the GNS construction is irreducible, and every irreducible representation is spatially equivalent to a cyclic representation associated with some pure state of $\mathfrak{A}$ [**13**, 3.13.2].

By (Stϕrmer [**14**, Theorem 5.1] and [**15**, Theorem 4.1]), a JW-algebra is a type I JW-factor if and only if it can be faithfully represented to act irreducibly on a complex Hilbert space. It is important to note that a concrete irreducible representation of a JC-algebra A is not necessarily dense. Indeed, let V be an infinite dimensional spin factor representation. We may suppose that V acts faithfully and irreducibly on a complex Hilbert space H. But then, $V \neq B(H)_{s.a}$, for otherwise, V would be universally reversible [**8**, 7.4.6]. Also, it is easy to see that a JC-algebra A acts irreducibly on a Hilbert space H if and only if its universal enveloping C*-algebra $C^*(A)$ acts irreducibly on H [**10**, Lemma 1].

An ideal (respectively, a Jordan ideal) I of a C*-algebra (respectively, a JC-algebra) $\mathfrak{A}$ is called **primitive** if it is the kernel of some irreducible representation (respectively, a type I factor representation) of $\mathfrak{A}$. **The primitive space**, $\Pr im(\mathfrak{A})$, of $\mathfrak{A}$ is the space of all primitive ideals of $\mathfrak{A}$. Given subsets $S \subseteq \mathfrak{A}$, $F \subseteq \Pr im(\mathfrak{A})$ we write $hull(S) = \{P \in \Pr im(\mathfrak{A}) : S \subseteq P\}$, $\ker(F) = \cap\{P : P \in F\}$. The class $\{hull(S) : S \subseteq \mathfrak{A}\}$ form the closed sets for a topology on $\Pr im(\mathfrak{A})$ called **the Jacobson or hull-kernel topology** (see [**13**, 4.1.1, 4.1.2] and [**3**, p. 2]). Since equivalent (respectively, Jordan equivalent) representations have the same kernels, $\Pr im(\mathfrak{A}) = \{\ker \pi_\rho : \rho \in P(\mathfrak{A})\}$ [**13**, 3.13.7], [**1**, Proposition 2.2]. The set of equivalence classes of all irreducible representations of a C*-algebra (respectively, type I factor representations of a JC-algebra) $\mathfrak{A}$ is denoted by $\sigma(\mathfrak{A})$, and is called **the primitive spectrum** of $\mathfrak{A}$. It is endowed with the inverse hull-kernel topology from $\Pr im(\mathfrak{A})$ induced by the natural surjection $g : \sigma(\mathfrak{A}) \to \Pr im(\mathfrak{A})$, given by $g([\pi]) = \ker \pi$. We define $\overset{\wedge}{hull}(J)$ in $\sigma(\mathfrak{A})$ as the preimage $g^{-1}(hull(J))$ of $hull(J)$ in $\Pr im(\mathfrak{A})$.

Lemma 1. *Let A be a JC-algebra, and ρ be a pure state of A. If I_i is a closed Jordan ideal of A such that $I_1 \cap I_2 \subseteq \ker \phi_\rho$, $i = 1, 2$, then either $I_1 \subseteq \ker \phi_\rho$ or $I_2 \subseteq \ker \phi_\rho$.*

Proof. Recall that $\phi_\rho : A \to c(\rho)A^{**}(a \mapsto c(\rho)a)$ is the type I factor representation of A and let $\bar{\phi}_\rho : A^{**} \to c(\rho)A^{**}(x \mapsto c(\rho)x)$ be its normal extension to A^{**} [**8**, 4.5.7]. Identifying the weak closure I_i^- of I_i in A^{**} with e_iA^{**}, for some central projection e_i of A^{**}, as in [**8**, 4.5.8], we have $I_1 \cap I_2 = e_1e_2A^{**} \cap A$ since $I_i = e_iA^{**} \cap A$. It follows that $I_1^- \cap I_2^- = e_1e_2A^{**} = (I_1 \cap I_2)^- \subseteq \ker \bar{\phi}_\rho$. If neither I_1 nor I_2 are contained in $\ker \phi_\rho$, then there is an element $a_i \in I_i$, such that $c(\rho)a_i = \phi_\rho(a_i) \neq 0$, which implies that $c(\rho)c(a_i) \neq 0$, where $c(a_i)$ is the central support of a_i in A^{**}. Since $c(\rho)c(a_i) \leq c(\rho)$ and $c(\rho)$ is a minimal central projection in A^{**}, $c(\rho)c(a_i) = c(\rho)$. It follows that $c(\rho)(c(a_1)c(a_2)) = c(\rho) \neq 0$, contradicting the fact that $c(a_1)c(a_2) \in Z(A^{**}) \subseteq Z(I_1^-) \cap Z(I_2^-) \subseteq \ker \bar{\phi}_\rho$. Hence, either $I_1 \subseteq \ker \phi_\rho$ or $I_2 \subseteq \ker \phi_\rho$. □

It is shown in [**3**, Proposition 2.1] that if A is a universally reversible JC-algebra, then the map $\psi : \Pr im(C^*(A)) \to \Pr im(A)$ given by $I \mapsto I \cap A$ is an open, closed and continuous surjection. Our next Theorem gives a similar relationship between the spectrum $\sigma(C^*(A))$ of $C^*(A)$ and the spectrum $\sigma(A)$ of A. But first recall that if A is a universally reversible JC-algebra and if $\mathfrak{J}$ is a Φ_A-invariant ideal of $C^*(A)$, then $\mathfrak{J}$ is generated by $\mathfrak{J} \cap A$, that is $\mathfrak{J} = [\mathfrak{J} \cap A]$ [**7**, Lemma 4.3].

Theorem 4. *Let A be a universally reversible JC-algebra. Then the map*

$$\Psi : \sigma(C^*(A)) \to \sigma(A), \quad ([\pi] \mapsto [\pi \mid_A])$$

is an open, closed and continuous surjection.

Proof. First note that if $\pi_i : C^*(A) \to B(H_i)$ is an irreducible representation of $C^*(A)$, $i = 1, 2$, such that π_1 and π_2 are equivalent, then $\pi_1 \mid_A$ and $\pi_2 \mid_A$ are Jordan equivalent irreducible representations of A, hence Ψ is well defined and clearly is a surjection, by [**14**, Theorem 5.1], [**15**, Theorem 4.1] and [**10**, Lemma 1].

Suppose that I is a closed Jordan ideal of A, and let $[I]$ be the closed ideal generated by I in $C^*(A)$. Recall the open continuous surjection maps $\widetilde{g} : \sigma(C^*(A)) \to \Pr im(C^*(A))$, $g : \sigma(A) \to \Pr im(A)$ $([\pi] \mapsto \ker \pi)$. If $[\pi] \in \sigma(C^*(A))\backslash \overset{\wedge}{hull}([I])$, that is $[\pi] \notin \overset{\wedge}{hull}([I]) = \widetilde{g}^{-1}(hull([I]))$, then $\ker \pi = \widetilde{g}([\pi]) \notin hull([I])$, which implies that $\ker \pi \not\supseteq [I]$, and so, $\ker \pi \cap A \not\supseteq [I] \cap A = I$. Hence $\ker \pi \mid_A = \ker \pi \cap A \notin hull(I)$, which implies that $[\pi \mid_A] \notin \overset{\wedge}{hull}(I)$, that is, $[\pi \mid_A] \in \sigma(A)\backslash \overset{\wedge}{hull}(I)$. On the other hand, if $[\pi \mid_A] \in \sigma(A)\backslash \overset{\wedge}{hull}(I)$, for some irreducible representation π of $C^*(A)$, then $\ker \pi \cap A = \ker \pi \mid_A \notin hull(I)$, which implies that $\ker \pi \not\supseteq [I]$, that is $\ker \pi \notin hull([I])$, and so $[\pi] \notin \overset{\wedge}{hull}([I])$. Therefore, $\Psi^{-1}\left(\sigma(A)\backslash \overset{\wedge}{hull}(I)\right) = \sigma(C^*(A))\backslash \overset{\wedge}{hull}([I])$ and Ψ is continuous.

Next, note that if $\mathfrak{J}$ is a closed ideal of $C^*(A)$ and $[\pi \mid_A] \in \Psi(\overset{\wedge}{hull}(\mathfrak{J}))$, for some $[\pi] \in \overset{\wedge}{hull}(\mathfrak{J}) = \widetilde{g}^{-1}(hull(\mathfrak{J}))$, then $\ker \pi \supseteq \mathfrak{J}$. It follows that, $g([\pi \mid_A]) = \ker \pi \mid_A = \ker \pi \cap A \supseteq \mathfrak{J} \cap A$ and so, $[\pi \mid_A] \in \overset{\wedge}{hull}(\mathfrak{J} \cap A)$. That is, $\Psi(\overset{\wedge}{hull}(\mathfrak{J})) \subseteq \overset{\wedge}{hull}(\mathfrak{J} \cap A)$. Conversely, let $[\pi \mid_A] \in \overset{\wedge}{hull}(\mathfrak{J} \cap A) = g^{-1}(hull(\mathfrak{J} \cap A))$, then
$g([\pi \mid_A]) = \ker \pi \mid_A = \ker \pi \cap A \supseteq \mathfrak{J} \cap A \in hull(\mathfrak{J} \cap A)$. Since $\mathfrak{J} \cap A = \Phi_A(\mathfrak{J}) \cap A$, and $\mathfrak{J} \cap \Phi_A(\mathfrak{J})$ is Φ_A-invariant, $[\mathfrak{J} \cap A] = \mathfrak{J} \cap \Phi_A(J)$, by [**7**, Lemma 4.3]. Therefore, $\ker \pi \supseteq [\ker \pi \mid_A] \supseteq [\mathfrak{J} \cap A] = \mathfrak{J} \cap \Phi_A(\mathfrak{J})$. But then $\ker \pi \supseteq \mathfrak{J}$ or $\ker \pi \supseteq \Phi_A(\mathfrak{J})$, because $\ker \pi$ is primitive [**13**, 3.13.9], so, $\mathfrak{J} \subseteq \ker \pi$ or $\mathfrak{J} \subseteq \Phi_A(\ker \pi)$, which implies that both $\ker \pi$ and $\Phi_A(\ker \pi)$ are contained in $hull(\mathfrak{J})$. Since $\Phi_A(\ker \pi) = \ker(\pi \circ \Phi_A)$ and π and $\pi \circ \Phi_A$ are obviously equivalent irreducible representations of $C^*(A)$, we have $[\pi] \in \widetilde{g}^{-1}(hull(\mathfrak{J})) = \overset{\wedge}{hull}(\mathfrak{J})$, and so, $[\pi \mid_A] = \Psi([\pi]) \in \Psi(\overset{\wedge}{hull}(\mathfrak{J}))$. That is, $\overset{\wedge}{hull}(\mathfrak{J} \cap A) \subseteq \Psi(\overset{\wedge}{hull}(\mathfrak{J}))$. Therefore, $\Psi(\overset{\wedge}{hull}(\mathfrak{J})) = \overset{\wedge}{hull}(\mathfrak{J} \cap A)$, and Ψ is a closed map.

It remains to prove that Ψ is open, so let $\mathfrak{J}$ be a closed ideal of $C^*(A)$. We claim that $\Psi\left(\sigma(C^*(A))\backslash \overset{\wedge}{hull}(\mathfrak{J})\right) = \sigma(A)\backslash \overset{\wedge}{hull}(\mathfrak{J} \cap A)$. Indeed, if $[\pi \mid_A] = \Psi([\pi]) \in \Psi\left(\sigma(C^*(A))\backslash \overset{\wedge}{hull}(\mathfrak{J})\right)$, then $\ker \pi \not\supseteq \mathfrak{J}$, which implies that $\Phi_A(\ker \pi) \not\supseteq \Phi_A(\mathfrak{J})$. But π and $\pi \circ \Phi_A$ are equivalent irreducible representations of $C^*(A)$ and so $\ker \pi = \ker(\pi \circ \Phi_A) = \Phi_A(\ker \pi)$. It follows that $\ker \pi$ does not contain the Φ_A-invariant closed ideal $\mathfrak{J} + \Phi_A(\mathfrak{J})$ [**12**, 10.1.9] of $C^*(A)$, and hence $\ker \pi \not\supseteq \mathfrak{J} + \Phi_A(\mathfrak{J}) = [(\mathfrak{J} + \Phi_A(\mathfrak{J})) \cap A] = [\mathfrak{J} \cap A]$. Therefore, $\ker \pi \mid_A \not\supseteq \mathfrak{J} \cap A$ which implies that $[\pi \mid_A] \notin \overset{\wedge}{hull}(\mathfrak{J} \cap A)$. That is, $[\pi \mid_A] \in \sigma(A)\backslash \overset{\wedge}{hull}(\mathfrak{J} \cap A)$. On the other hand, if $[\pi \mid_A] \notin \overset{\wedge}{hull}(\mathfrak{J} \cap A)$, then $\ker \pi \cap A = \ker \pi \mid_A \not\supseteq \mathfrak{J} \cap A$ which implies that $\ker \pi \not\supseteq \mathfrak{J}$.

Therefore, $\widetilde{g}([\pi]) = \ker\pi \notin hull(\mathfrak{J})$, and so, $[\pi] \notin \widetilde{g}^{-1}(hull(\mathfrak{J})) = \widehat{hull}(\mathfrak{J})$. It follows that $[\pi\mid_A] = \Psi([\pi]) \in \Psi\left(\sigma(C^*(A))\backslash\widehat{hull}(\mathfrak{J})\right)$. Hence,

$$\Psi\left(\sigma(C^*(A))\backslash\widehat{hull}(\mathfrak{J})\right) = \sigma(A)\backslash\widehat{hull}(\mathfrak{J}\cap A), \text{ proving that } \Psi \text{ is open.} \qquad \square$$

3. Tensor Product of States and Representations of JC-algebras

In this section we give the Jordan analogue of (Guichardet [**5**, Proposition 5]) which asserts that if $\mathfrak{A}_i, i = 1,2$ is a C*-algebra, then the mappings $(\rho_1,\rho_2) \mapsto \rho_1 \underset{\max}{\otimes} \rho_2$ of $S(\mathfrak{A}_1)\times S(\mathfrak{A}_2)$ into $S(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2)$ and $(\pi_1,\pi_2) \mapsto \pi_1 \underset{\max}{\otimes} \pi_2$ of $\sigma(\mathfrak{A}_1)\times\sigma(\mathfrak{A}_2)$ into $\sigma(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2)$ are continuous, where $\rho_1 \underset{\max}{\otimes} \rho_2$ is **the product state of ρ_1 and ρ_2** (see [**12**, 11.3.8], [**10**, Theorem 6]). Then we establish the Jordan analogue of the following commutative diagram

$$\begin{array}{ccc} P(\mathfrak{A}_1)\times P(\mathfrak{A}_2) & \underset{\rightarrow}{\theta} & P(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2) \\ T'\downarrow & & \downarrow T \\ \sigma(\mathfrak{A}_1)\times\sigma(\mathfrak{A}_2) & \underset{\rightarrow}{\Phi} & \sigma(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2) \end{array}$$

[**5**, P. 14], where T is the canonical map $(\rho \mapsto \pi_\rho) : P(\mathfrak{A}) \to \sigma(\mathfrak{A})$, which is known to be open, continuous and surjective [**4**, 3.4.11], and π_ρ is the irreducible representation associated with ρ by the GNS construction [**11**, 4.5.2], [**17**, I.9.14].

For clarity and convenience we prove the following Theorem which is used in [**5**, P. 13] without proof.

Theorem 5. *If $\{\pi_i, H_i\}$ is a representation of a C*-algebra $\mathfrak{A}_i, i = 1,2$, then $\pi_1 \underset{\max}{\otimes} \pi_2$ is irreducible.*

Proof. Since the identity map on $\mathfrak{A}_1\otimes\mathfrak{A}_2$ extends to a surjective *-homomorphism $\psi : \mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2 \to \mathfrak{A}_1 \underset{\min}{\otimes} \mathfrak{A}_2$, the following diagram commutes

$$\begin{array}{ccc} \mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2 & \overset{\psi}{\rightarrow} & \mathfrak{A}_1 \underset{\min}{\otimes} \mathfrak{A}_2 \\ {\scriptstyle \pi_1 \underset{\max}{\otimes} \pi_2}\searrow & & \swarrow{\scriptstyle \pi_1 \underset{\min}{\otimes} \pi_2} \\ & B(H_1\otimes H_2) & \end{array}$$

Hence

$$\begin{aligned} (\pi_1 \underset{\max}{\otimes} \pi_2)(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2) &= ((\pi_1 \underset{\min}{\otimes} \pi_2)\circ\psi)(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2) \\ &= (\pi_1 \underset{\min}{\otimes} \pi_2)(\mathfrak{A}_1 \underset{\min}{\otimes} \mathfrak{A}_2), \end{aligned}$$

so it follows from the commutation theorem [**12**, 11.2.6] that

$$((\pi_1 \underset{\max}{\otimes} \pi_2)(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2))' = ((\pi_1 \underset{\min}{\otimes} \pi_2)(\mathfrak{A}_1 \underset{\min}{\otimes} \mathfrak{A}_2))' = (\pi_1(\mathfrak{A}_1))' \,\bar{\otimes}\, (\pi_2(\mathfrak{A}_2))'.$$

Therefore, $((\pi_1 \underset{\max}{\otimes} \pi_2)(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2))'$ consists of scalars if and only if $(\pi_i(\mathfrak{A}_i))'$ consists of scalars. Hence, $\pi_1 \underset{\max}{\otimes} \pi_2$ is irreducible by [**13**, 3.13.2]. $\square$

Theorem 6. *Let $\mathfrak{A}_i$ be a C^*-algebra, H_i and K_i be Hilbert spaces, $i = 1, 2$. If $\pi_i : \mathfrak{A}_i \to B(H_i)$ and $\phi_i : \mathfrak{A}_i \to B(K_i)$ are representations of $\mathfrak{A}_i$ such that π_i is spatially equivalent to ϕ_i, then $\pi_1 \underset{\max}{\otimes} \pi_2$ is spatially equivalent to $\phi_1 \underset{\max}{\otimes} \phi_2$.*

Proof. Let $u_i : H_i \to K_i, i = 1, 2$, be the isometries that effectuate the equivalence, i.e. $u_i\pi_i(x_i)u_i^* = \phi_i(x)$ for all $x_i \in \mathfrak{A}_i$. Then $u_1 \otimes u_2 : H_1 \otimes H_2 \to K_1 \otimes K_2$ given by $(u_1 \otimes u_2)(\xi_1 \otimes \xi_2) = u_1(\xi_1) \otimes u_2(\xi_2)$ is an isometry [**11**, 2.6.12]. Thus,

$$\begin{aligned}(u_1 \otimes u_2)(\pi_1 \otimes \pi_2)(x_1 \otimes x_2)(u_1^* \otimes u_2^*) &= u_1\pi_1(x_1)u_1^* \otimes u_2\pi_2(x_2)u_2^* \\ &= \phi_1(x) \otimes \phi_2(x), \quad \forall x_i \in \mathfrak{A}_i.\end{aligned}$$

The proof is completed by linearity and continuity of $\pi_1 \underset{\max}{\otimes} \pi_2$ and $\phi_1 \underset{\max}{\otimes} \phi_2$. □

Remark 1. *Note that if ρ_i is a pure state of a C^*-algebra A_i, $i = 1, 2$, then $\rho_1 \underset{\max}{\otimes} \rho_2$ is pure [5, p.13]. Indeed, since the cyclic representation $\{\pi_{\rho_i}, H_{\rho_i}, \xi_{\rho_i}\}$ associated with ρ_i by the GNS construction is irreducible [17, I.9.22], $\pi_{\rho_1} \underset{\max}{\otimes} \pi_{\rho_2}$ is irreducible by Theorem 5. If $x_i \in A_i$, $i = 1, 2$, then*

$$\begin{aligned}(\rho_1 \underset{\max}{\otimes} \rho_2)(x_1 \otimes x_2) &= \rho_1(x_1)\rho_2(x_2) \\ &= \langle \pi_{\rho_1}(x_1)\xi_{\rho_1}, \xi_{\rho_1}\rangle\langle \pi_{\rho_2}(x_2)\xi_{\rho_2}, \xi_{\rho 2}\rangle \\ &= \langle (\pi_{\rho_1} \otimes \pi_{\rho_2})(x_1 \otimes x_2)(\xi_{\rho_1} \otimes \xi_{\rho_2}), \xi_{\rho_1} \otimes \xi_{\rho_2}\rangle.\end{aligned}$$

By linearity and continuity of $\pi_{\rho_1} \underset{\max}{\otimes} \pi_{\rho_2}$, and continuity of inner product, we have

$$(\rho_1 \underset{\max}{\otimes} \rho_2)(z) = \langle (\pi_{\rho_1} \underset{\max}{\otimes} \pi_{\rho_2})(z)(\xi_{\rho_1} \otimes \xi_{\rho_2}), \xi_{\rho_1} \otimes \xi_{\rho_2}\rangle, \quad \forall z \in \mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2.$$

*Therefore, $\pi_{\rho_1} \underset{\max}{\otimes} \pi_{\rho_2}$ is spatially equivalent to the representation $\pi_{\rho_1 \underset{\max}{\otimes} \rho_2}$ [**11**, 4.5.3], which implies that $\rho_1 \underset{\max}{\otimes} \rho_2$ is pure, by [**17**, I.9.22].*

Definition 2. *A JC-algebra A is said to be* ***of complex type*** *if all its type I factor representations are into a type I JW-factor isomorphic to $B(H)_{s.a}$ for some complex Hilber space H. That is, if $\phi : A \to M$ is a type I factor representation, where M is a type I JW-factor, then $\phi(A)^- = M \cong B(H)_{s.a}$.*

Remark 2. *(i) Let $A = \mathfrak{A}_{s.a}$, where $\mathfrak{A}$ is a C^*-algebra, and let $\phi : A \to M$ be a type I factor representation of A. Then ϕ extends to a normal surjection $\bar{\phi} : A^{**} \to M$, by [**8**, 4.5.7]. Thus, there exists a central projection e in A^{**} such that $eA^{**} \cong M$. Since A^{**} is the self-adjoint part of a von Neumann algebra, M is isomorphic to the self-adjoint part of a type I von Neumann factor. Hence $M \cong B(H)_{s.a}$, for some Hilbert space H, by [**8**, 7.5.2].*

*(ii) If A is a JC-algebra, then A is of complex type if and only if all its concrete irreducible representations are dense [**1**, Proposition 3.5]. Indeed, if A is of complex type, and $\pi : A \to B(H)$ is an irreducible representation of A, then $\pi(A)^-$ is an irreducible JW-algebra, and so it is a type I JW-factor [**15**, Theorem 4.1]. Since A is of complex type, $\pi(A)^-$ is the self adjoint part of a von Neumann algebra, and so $\pi(A)^- = B(H)_{s.a}$. The converse is obvious by [**14**, Theorem 5.1].*

Lemma 2. *Let A_i be a JC-algebra of complex type, H_i and K_i be Hilbert spaces, $i = 1, 2$. If $\pi_i : A_i \to B(H_i)_{s.a}$ and $\phi_i : A_i \to B(K_i)_{s.a}$ are irreducible representations of A_i such that π_i is Jordan equivalent to ϕ_i, then $\pi_1 \underset{\max}{\otimes} \pi_2$ is Jordan equivalent to $\phi_1 \underset{\max}{\otimes} \phi_2$.*

Proof. Since A_i is of complex type and π_i is Jordan equivalent to ϕ_i, $B(H_i)_{s.a} \cong B(K_i)_{s.a}$ which implies that $B(H_i) \cong B(K_i)$. Therefore, $\overset{\wedge}{\pi_i} : C^*(A_i) \to B(H_i)$ and $\overset{\wedge}{\phi_i} : C^*(A_i) \to B(K_i)$ are equivalent irreducible representations of $C^*(A_i)$ and hence they are spatially equivalent, by [**12**, 10.3.7]. It follows that $\overset{\wedge}{\pi_1} \underset{\max}{\otimes} \overset{\wedge}{\pi_2}$ is spatially equivalent to $\overset{\wedge}{\phi_1} \underset{\max}{\otimes} \overset{\wedge}{\phi_2}$, which implies that $\overset{\wedge}{\pi_1} \underset{\max}{\otimes} \overset{\wedge}{\pi_2}$ is equivalent to $\overset{\wedge}{\phi_1} \underset{\max}{\otimes} \overset{\wedge}{\phi_2}$ and so, $\pi_1 \underset{\max}{\otimes} \pi_2 = (\overset{\wedge}{\pi_1} \underset{\max}{\otimes} \overset{\wedge}{\pi_2}) \mid_{JC(A_1 \underset{\max}{\otimes} A_2)}$ is Jordan equivalent to $\phi_1 \underset{\max}{\otimes} \phi_2 = (\overset{\wedge}{\phi_1} \underset{\max}{\otimes} \overset{\wedge}{\phi_2}) \mid_{JC(A_1 \underset{\max}{\otimes} A_2)}$. □

Lemma 3. *Every JC-algebra of complex type is universally reversible.*

Proof. Let A be a JC-algebra of complex type, and let $\phi : A \to V$ be a spin factor representation of A. Since A is of complex type $\phi(A)^- = V \cong B(H)_{s.a}$, for some complex Hilber space H, which implies that V is universally reversible, by [**8**, 7.4.6], and so, $V = V_2$ or $V = V_3$, by [**1**, p.280]. Therefore, all the spin factor representations of A are onto V_2 or V_3. That is, A is universally reversible by [**7**. Prpopsition 2.2].. □

Definition 3. *Let A be a JC-algebra of complex type and let ρ be a pure state of A. An irreducible concrete representation $\pi : A \to B(H)_{s.a}$ is said to be* ***associated with*** *ρ if there is a unit vector ξ such that $\rho(a) = \langle \pi(a)\xi, \xi\rangle$ for all $a \in A$.*

Note that the unit vector in the above definition is uniquely determined up to a scalar multiple of modulus one by virtue of the density of $\pi(A)$ in $B(H)_{s.a}$.

Lemma 4. *Let A be a JC-algebra and let $\phi : A \to M \subseteq B(H)_{s.a}$ be a type I factor of A. Then there is a unit vector $\xi \in H$ such that $\omega_\xi \circ \phi$ is a pure state of A.*

Proof. Let $\bar{\phi} : A^{**} \to M$, the normal extension of ϕ to A^{**}. Then there exists a central projection e in A^{**} such that $eA^{**} \cong \bar{\phi}(A^{**}) = M$, hence the restriction ψ of $\bar{\phi}$ to eA^{**} is an isomorphism of eA^{**} onto M. Since M is a JW-factor of type I, there is a unit vector $\xi \in H$ such that the vector state ω_ξ on M is pure [**15**, Theorem 4.3], and so, $\omega_\xi \circ \psi$ is a pure state of eA^{**}. Since $(1-e)A^{**} \subseteq \ker \omega_\xi \circ \bar{\phi}$ and $A^{**} = eA^{**} \oplus (1-e)A^{**}$, it is now easy to see that $\omega_\xi \circ \bar{\phi}$ is a pure state of A^{**} which is normal. Since a normal state ρ on A^{**} is pure if and only if $\rho \mid_A$ is pure on A, we see that $\omega_\xi \circ \phi = (\omega_\xi \circ \bar{\phi}) \mid_A$ is a pure state of A. □

Remark 3. *Let A be a JC-algebra of complex type.*

(i) If ρ is a pure state of A and $\{\pi, H, \xi\}$ is the cyclic representation associated with $\hat{\rho}$ obtained by the GNS construction, where $\hat{\rho}$ is a pure state extension of ρ to $C^*(A)$, then $\{\pi \mid_A, H, \xi\}$ is an irreducible concrete representation associated with ρ.

*(ii) If $\pi : A \to B(H)$ is an irreducible representation associated with a pure state ρ of A, then π is a type I factor representation and hence π is Jordan equivalent to ϕ_υ for some $\upsilon \in P(A)$ and so $\pi^* : B(H)^* \to A^*$ maps the set S of normal states of $B(H)$ injectively onto the smallest split face F_υ of $S(A)$ containing υ [**1**, Proposition 2.2]. Since π is associated with ρ*

$$\rho(a) = \langle \pi(a)\xi, \xi \rangle = (\pi^*(\omega_\xi))(a)\ ,\ a \in A,$$

*for some unit vector $\xi \in H$. Thus $\rho = \pi^*(\omega_\xi) \in \pi^*(S) = F_\upsilon$, so $F_\rho \subseteq F_\upsilon$. Since F_υ is a minimal split face, $F_\rho = F_\upsilon$ which implies that ϕ_υ (and hence π) is Jordan equivalent to ϕ_ρ [**1**, p. 273].*

(iii) If $\rho_1, \rho_2 \in P(C^*(A))$ such that $\rho = \rho_1 \mid_A = \rho_2 \mid_A$ and $\{\pi_1, H_1, \xi_1\}$, $\{\pi_2, H_2, \xi_2\}$ are the cyclic representations associated with ρ_1, ρ_2, respectively, then for each $a \in A$

$$\langle \pi_1(a)\xi_1, \xi_1 \rangle = \rho_1(a) = \rho(a) = \rho_2(a) = \langle \pi_2(a)\xi_2, \xi_2 \rangle.$$

Hence $\pi_1 \mid_A$ and $\pi_2 \mid_A$ are both associated with ρ which implies that $\pi_1 \mid_A$ is Jordan equivalent to $\pi_2 \mid_A$, by (ii).

The following Theorem is the Jordan analogue of [**4**, 3.4.11].

Theorem 7. *Let A be a JC-algebra of complex type and let $T_\circ : P(A) \to \sigma(A)$ be the map given by $\rho \mapsto [\phi_\rho]$. Then $T_\circ$ is an open continuous surjection.*

Proof. It is clear from Remark 3(iii) that $T_\circ$ is well defined, and is obviously surjective. To see that $T_\circ$ is an open map, note that if $\rho \in P(C^*(A))$ then $\rho \mid_A \in P(A)$, since A is of complex type [**1**, Proposition 5.3]. So, let r be the restriction map $(\psi_A)^* \mid_{P(C^*(A))}$ from $P(C^*(A))$ onto $P(A)$, where $(\psi_A)^* : C^*(A)^* \to A^*$ is the dual of the embedding $\psi_A : A \to C^*(A)$, and consider the following commutative diagram.

$$\begin{array}{ccc} P(C^*(A)) & \underrightarrow{T} & \sigma(C^*(A)) \\ r \downarrow & & \downarrow \Psi \\ P(A) & \underrightarrow{T_\circ} & \sigma(A) \end{array}$$

The fact that $T_\circ$ is open follows immediately, since T, Ψ and r are all surjective maps, T and Ψ are open and r is continuous. It remains to prove that $T_\circ$ is continuous, so let $\eta : P(A) \to \Pr im(A)$ $(\rho \longmapsto \ker \phi_\rho)$. By [**6**, Theorem 4.1], η is a continuous, open and surjective map. Since $g : \sigma(A) \to \Pr im(A)$ is open and continuous surjection (see [**13**, 4.1.12]), it is easy now to conclude from the following commutative diagram

$$\begin{array}{ccc} P(A) & \underrightarrow{T_\circ} & \sigma(A) \\ {}_{\eta}\searrow & & \swarrow_{g} \\ & \Pr im(A) & \end{array}$$

that $T_\circ$ is also continuous, completing the proof. $\square$

Let $\mathfrak{A}_i$ be a C*-algebra, $i = 1, 2$. Then the map

$$([\pi_1], [\pi_2]) \mapsto [\pi_1 \underset{\max}{\otimes} \pi_2] : \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2) \to \sigma(\mathfrak{A}_1 \underset{\max}{\otimes} \mathfrak{A}_2)$$

is continuous. The following Theorem gives the Jordan analogue of this result.

Theorem 8. *If A_i, $i = 1, 2$, is a JC-algebra of complex type, then the map $\Phi_\circ : \sigma(A_1) \times \sigma(A_2) \to \sigma(JC(A_1 \underset{\max}{\otimes} A_2))$ given by $([\pi_1], [\pi_2]) \mapsto [\pi_1 \underset{\max}{\otimes} \pi_2]$, where $[\pi_i] \in \sigma(A_i)$, is continuous.*

Proof. By Lemma 2, if $\pi_i : A_i \to B(H_i)_{s.a}$ and $\phi_i : A_i \to B(K_i)_{s.a}$ are irreducible representations of A_i such that π_i is Jordan equivalent to ϕ_i, then $\pi_1 \underset{\max}{\otimes} \pi_2$ is Jordan equivalent to $\phi_1 \underset{\max}{\otimes} \phi_2$, hence $\Phi_\circ$ is well defined. To show that $\Phi_\circ$ is continuous, let $\Phi : \sigma(C^*(A_1)) \times \sigma(C^*(A_2)) \to \sigma(C^*(A_1) \underset{\max}{\otimes} C^*(A_2))$ be the continuous map given in [**5**, Proposition 5], then consider the following commutative diagram

$$\begin{array}{ccc} \sigma(C^*(A_1)) \times \sigma(C^*(A_2)) & \xrightarrow{\Phi} & \sigma(C^*(A_1) \underset{\max}{\otimes} C^*(A_2)) \\ \Psi' \downarrow & & \downarrow \Psi \\ \sigma(A_1) \times \sigma(A_2) & \xrightarrow{\Phi_\circ} & \sigma(JC(A_1 \underset{\max}{\otimes} A_2)) \end{array}$$

from which it is easy to conclude that $\Phi_\circ$ is continuous, since Ψ and Ψ' are surjective, continuous and open maps (cf. Theorem 4), and Φ is continuous. □

Theorem 9. *Let A_i, $i = 1, 2$, be a JC-algebra of complex type, then the map $\theta_\circ : P(A_1) \times P(A_2) \to P(JC(A_1 \underset{\max}{\otimes} A_2))$ given by $(\rho_1, \rho_2) \mapsto \rho_1 \underset{\max}{\otimes} \rho_2$ is continuous.*

Proof. Let $\rho_i \in P(A_i)$, we shall first show that $\rho_1 \underset{\max}{\otimes} \rho_2$ is a pure state of $JC(A_1 \underset{\max}{\otimes} A_2)$. So let $\hat{\rho}_i$ be an extention of ρ_i to $C^*(A_i)$, then $\hat{\rho}_i$ is pure and hence the cyclic representation $(\hat{\pi}_i, H_i, \xi_i)$ associated with $\hat{\rho}_i$ is irreducible, by [**17**, I.9.22]. If π_i is the restriction of $\hat{\pi}_i$ to A_i, then π_i is irreducible [**10**, Lemma 1]. Therefore $\pi_1 \underset{\max}{\otimes} \pi_2 = (\hat{\pi}_1 \underset{\max}{\otimes} \hat{\pi}_2) \mid_{JC(A_1 \underset{\max}{\otimes} A_2)}$ is irreducible, by Theorem 5 and [**10**, Lemma 1] and so the weak closure $\left((\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))\right)^-$ of $(\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))$ in $B(H_1 \otimes H_2)$ is a type I JW-factor. Note that $\hat{\rho}_i = \omega_{\xi_i} \circ \hat{\pi}_i$, where ω_{ξ_i} is the vector state $x \mapsto \langle x\xi_i, \xi_i\rangle$ which is a pure state of $B(H_i)$ (see [**12**, 10.2.5]). Since A_i is of complex type, $\pi_i(A_i)^- = B(H_i)_{s.a}$ and hence the restriction of ω_{ξ_i} to $\pi_i(A_i)^-$ is a pure state (cf. Remark 2(ii)). Therefore, the support e_i of ω_{ξ_i} in $\pi_i(A_i)^-$ is a minimal (and hence abelian) projection [**15**, Corollary 4.4]. Since $\pi_1(A_1) \otimes 1 \hookrightarrow (\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))$ and $1 \otimes \pi_2(A_2) \hookrightarrow (\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))$, we have $\pi_1(A_1)^- \otimes 1 \hookrightarrow \left((\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))\right)^-$ and $1 \otimes \pi_2(A_2)^- \hookrightarrow \left((\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))\right)^-$. Hence $\left((\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))\right)^-$ contains the JW-algebra generated by the JW-algebra $\pi_1(A_1)^- \otimes 1$ and the JW-algebra $1\otimes\pi_2(A_2)^-$, which implies that $e_1\otimes e_2$ belongs to $\left((\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))\right)^-$. Since $\pi_1(A_1) \otimes \pi_2(A_2) = (\pi_1 \underset{\max}{\otimes} \pi_2)(A_1 \otimes A_2) \hookrightarrow (\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))$, $\left((\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))\right)^-$ is the JW-algebra generated by $\pi_1(A_1)^-\otimes\pi_2(A_2)^-$. By [**18**, III.5.12], $e_1 \otimes e_2$ is the support of $\omega_{\xi_1} \otimes \omega_{\xi_2} = \omega_{\xi_1\otimes\xi_2}$, and is obviously abelian since $(e_1 \otimes e_2)B(H_1 \otimes H_2)(e_1 \otimes e_2)$ is unitarily equivalent to $e_1B(H_1)e_1 \bar{\otimes} e_2B(H_2)e_2$. Therefore, $\omega_{\xi_1\otimes\xi_2}$ is a pure state of $\left((\pi_1 \underset{\max}{\otimes} \pi_2)(JC(A_1 \underset{\max}{\otimes} A_2))\right)^-$, by [**15**, Corollary 4.4]. Note that for all $a_i \in A_i$

$$\begin{aligned}
(\rho_1 \underset{\max}{\otimes} \rho_2)(a_1 \otimes a_2) &= \rho_1(a_1)\rho_2(a_2) \\
&= \langle \pi_1(a_1)\xi_1, \xi_1\rangle\langle \pi_2(a_2)\xi_2, \xi_2\rangle \\
&= \omega_{\xi_1\otimes\xi_2} \circ (\pi_1 \underset{\max}{\otimes} \pi_2)(a_1 \otimes a_2),
\end{aligned}$$

and hence by the uniqueness of $\rho_1 \underset{\max}{\otimes} \rho_2$, we must have $\rho_1 \underset{\max}{\otimes} \rho_2 = \omega_{\xi_1\otimes\xi_2}\circ(\pi_1 \underset{\max}{\otimes} \pi_2)$, which implies that $\rho_1 \underset{\max}{\otimes} \rho_2$ is a pure state of $JC(A_1 \underset{\max}{\otimes} A_2)$, and that $\theta_\circ$ is well defined.

The continuity of $\theta_\circ$ is immediate, since $\theta_\circ$ is the restriction of the map $S(A_1) \times S(A_2) \to S(JC(A_1 \underset{\max}{\otimes} A_2))$ $((\rho_1, \rho_2) \mapsto (\rho_1 \underset{\max}{\otimes} \rho_2)$ to $P(A_1) \times P(A_2)$ and the continuity of this map follows easily from the continuity of $\theta : S(C^*(A_1)) \times S(C^*(A_2)) \mapsto S(C^*(A_1) \underset{\max}{\otimes} C^*(A_2))$ $((f_1, f_2) \mapsto (f_1 \underset{\max}{\otimes} f_2)$ [**5**, Proposition 5] by Theorem 1 and since each state of A_i extends to a state of $C^*(A_i)$ and each state of $C^*(A_i)$ restricts to a state of A_i. □

Gathering the results of Theorem 7 and Theorem 9 we have

Theorem 10. *Let A_i, $i = 1, 2$, be a JC-algebra of complex type, then the following diagram of continuous maps is commutative.*

$$\begin{array}{ccc}
P(A_1) \times P(A_2) & \overset{\theta_\circ}{\longrightarrow} & P(JC(A_1 \underset{\max}{\otimes} A_2)) \\
T_\circ^{'} \downarrow & & \downarrow T_\circ \\
\sigma(A_1) \times \sigma(A_2) & \overset{\Phi_\circ}{\longrightarrow} & \sigma(JC(A_1 \underset{\max}{\otimes} A_2))
\end{array}$$

Corollary 1. *Let A_i, $i = 1, 2$, be a JC-algebra of complex type, then the following diagram is commutative.*

$$\begin{array}{ccccccc}
 & P(C^*(A_1)) \times P(C^*(A_2)) & & \underrightarrow{\theta} & & P(C^*(A_1) \underset{\max}{\otimes} C^*(A_2)) & \\
 & & {}^{r'}\searrow & & \swarrow_{r} & & \\
T^{'} \downarrow & & P(A_1) \times P(A_2) & \underrightarrow{\theta_\circ} & P(JC(A_1 \underset{\max}{\otimes} A_2)) & & \downarrow T \\
 & & T^{'}_{\circ} \downarrow & & \downarrow T_\circ & & \\
 & & \sigma(A_1) \times \sigma(A_2) & \underrightarrow{\Phi_\circ} & \sigma(JC(A_1 \underset{\max}{\otimes} A_2)) & & \\
 & & \Psi^{'} \nearrow & & \nwarrow \Psi & & \\
 & \sigma(C^*(A_1)) \times \sigma(C^*(A_2)) & & \underrightarrow{\Phi} & & \sigma(C^*(A_1) \underset{\max}{\otimes} C^*(A_2)) &
\end{array}$$

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