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Colocality and twisted sums of Banach spaces

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Abstract

Using the relation between subspaces of Banach spaces and quotients of their duals, we introduce the concept of colocality to give a new method that guarantees the existence of nontrivial twisted sums in which finite quotients play a major role (Theorem 1.7). An interesting point is that no restrictions are imposed on the quotients, only on the various subspaces. New examples of nontrivial twisted sums are given.

Keywords: Banach space; Sequence space; Locally bounded space; Complemented subspace; Three Space problem; Operator

0. Introduction

A **short exact sequence** is a diagram $0 \to Y \xrightarrow{i} X \xrightarrow{q} Z \to 0$ of quasi Banach spaces and bounded linear operators such that the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem implies that X contains $i(Y)$ and the quotient $X/i(Y)$ is isomorphic to Z. We shall also say that X is a **twisted sum** of Y and Z (or that X is an **extension of Y by Z**). The twisted sum X is said to be **trivial** if $i(Y)$ is complemented in X; otherwise, X is **nontrivial**.

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Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are said to be **equivalent** if there is a bounded linear operator T making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

commutative. The three lemma and the open mapping theorem imply that T must be an isomorphism, Cabello and Castillo [2, p. 525]. An exact sequence $0 \to Y \to X \to Z \to 0$ is said to **split** if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. We denote by $\mathrm{Ext}(Z, Y)$ the space of all equivalence classes of locally convex twisted sums of Y and Z. Thus $\mathrm{Ext}(Z, Y) = 0$ means that all locally convex twisted sums of Y and Z are equivalent to the direct sum $Y \oplus Z$. An operator $T : X \to Y$ of Banach spaces is an **isomorphism** if it is an invertible bounded linear map, T is an **isometry** if $\|Tx\| = \|x\|$ for every $x \in X$, it is a **λ-isomorphism**, $\lambda > 1$, if T is an isomorphism and $\|T\| < \lambda$, $\|T^{-1}\| < \lambda$, Heinrich [8, II.6]. We denote by $BL(X, Y)$ the space of all bounded linear maps of X into Y. The **distance** between two homogeneous maps T_1 and T_2 acting between the same spaces is given by

$$\mathrm{dist}(T_1, T_2) = \sup\bigl\{\|T_1x - T_2x\|\colon \|x\| \leqslant 1\bigr\}.$$

We note that bounded maps are those maps at finite distance from the zero map, also it should be kept in mind that linear maps are not assumed to be bounded. A homogeneous map $F : Z \to Y$ acting between two Banach spaces is said to be **z-linear** if it satisfies, for some constant k and all $z_i \in Z$,

$$\left\| F\left(\sum_{i=1}^{n} z_i\right) - \sum_{i=1}^{n} F(z_i) \right\| \leqslant k\left(\sum_{i=1}^{n} \|z_i\|\right).$$

The infimum of the constants k satisfying the above inequality is denoted by $Z(F)$ and is called the **z-linearity constant** of F. Every z-linear map is quasi-linear, however, the converse is not true (cf. [1, p. 7]). Given a quasi-linear map $F : Z \to Y$, it is possible to construct a twisted sum of Y and Z, denoted by $Y \oplus_F Z$, endowing the product space $Y \times Z$ with the quasi-norm $\|(y, z)\| = \|y - F(z)\| + \|z\|$. On the other hand, given a short exact sequences $0 \to Y \to X \to Z \to 0$, a quasi-linear map $F : Z \to Y$ can be obtained such that X is equivalent to $Y \oplus_F Z$, Kalton and Peck [11, Theorem 2.4 and p. 5]. A quasi-linear map $F : Z \to Y$ is said to be **trivial** if the twisted sum $Y \oplus_F Z$ is trivial, equivalently, F is at finite distance from a linear map $L : Z \to Y$, Kalton [10, Proposition 3.3]. Two quasi-linear maps F and G of a Banach space Z into a Banach space Y are said to be **equivalent** if the corresponding exact sequences $0 \to Y \to Y \oplus_F Z \to Z \to 0$ and $0 \to Y \to Y \oplus_G Z \to Z \to 0$ are equivalent. A quasi-linear map $F : Z \to Y$ is z-linear if and only if the twisted sum $Y \oplus_F Z$ is locally convex, Cabello and Castillo [1, Theorem 2] (see also Castillo and González [4, 1.6.e]).

Throughout this paper $\mathcal{E}$ and $\mathcal{F}$ denote families of finite dimensional Banach spaces and $\mathcal{E}^*$ denotes the family of the duals of the spaces in $\mathcal{E}$.

1. Colocality and twisted sums

Given a family $\mathcal{E}$ of finite dimensional Banach spaces, a Banach space X is said to **contain $\mathcal{E}$ uniformly complemented** if there exists a constant c such that for every $E \in \mathcal{E}$, there is a c-complemented subspace A of X which is c-isomorphic to E. It is clear that X contains $\mathcal{E}$

uniformly complemented if and only if its second dual X^{**} does. A Banach space X is said to be **λ-locally** $\mathcal{E}$ (or, if no quantitative is needed, **locally** $\mathcal{E}$) if there exists a constant $\lambda > 1$ such that every finite dimensional subspace A of X is contained in a finite dimensional subspace B of X such that $d_{BM}(B, E) < \lambda$, for some $E \in \mathcal{E}$, where $d_{BM}(B, E)$ is the **Banach–Mazur distance** between B and E, and is defined by $d_{BM}(B, E) = \inf\{\|T\|\|T^{-1}\|;\ T : X \to Y$ is an isomorphism of X onto $Y\}$, Cabello and Castillo [2]. If $\mathcal{E} = \{\ell_p^n\}_{n=1}^{\infty}$, then X is an $\boldsymbol{\mathcal{L}_p}$**-space**, Lindenstrauss and Rosenthal [12]. The locality of a family is used to determine the existence of nontrivial twisted sums of certain Banach spaces, in fact, if Y is a Banach space complemented in its bidual, and if all the locally convex twisted sums of Y and some Banach space W containing a family $\mathcal{E}$ uniformly complemented are trivial, then $\mathrm{Ext}(Z, Y) = 0$ for every Banach space Z which is locally $\mathcal{E}$, Cabello and Castillo [2, Theorem 2]. Using this fact, it is shown that there is a nontrivial twisted sums of ℓ_1 and ℓ_2, of ℓ_2 and c_0, and that $\mathrm{Ext}(c_0, \ell_1) \neq 0$, Cabello and Castillo [2, Examples 4.1, 4.2 and 4.3].

Since subspaces of Banach spaces are related to quotients of their duals, and vice versa, as we recall that if $G = X/D$ is a quotient of a Banach space X, then $G^* = (X/D)^* = D^{\perp}$ is a subspace of its dual X^*, and if B is w^*-closed in X^* then $B = (B_{\perp})^{\perp} = (X/B_{\perp})^*$ so that $B^* = (X/B_{\perp})^{**} = X/B_{\perp}$ is a quotient of X, where $D^{\perp}$ and $B_{\perp}$ denote the annihilators of D and B (see Rudin [13, 4.7, 4.8]), it is then natural to introduce an analogous notion of locality which involves quotients, and then study the existence of nontrivial twisted sums. We say that a Banach space X is **λ-colocally** $\boldsymbol{\mathcal{E}}$ (or **colocally** $\boldsymbol{\mathcal{E}}$) if there exists a constant $\lambda > 1$ such that every finite dimensional quotient A of X is a quotient of another finite dimensional quotient B of X satisfying $d_{BM}(B, E) < \lambda$ for some $E \in \mathcal{E}$. For example, the space $L_p(\mu)$, for any measure μ, is both locally and colocally $\{\ell_p^n\}_{n=1}^{\infty}$ (cf. Corollary 1.2).

Theorem 1.1. *Let X be a Banach space, then*:

(i) *X is λ-colocally $\mathcal{E}$ if and only if X^* is λ-locally $\mathcal{E}^*$.*
(ii) *If X^{**} is λ-locally $\mathcal{E}$ (respectively λ-colocally $\mathcal{E}$) then X is c-locally $\mathcal{E}$ (respectively c-colocally $\mathcal{E}$) for every $c > \lambda$.*

Proof. (i) Suppose that X is λ-colocally $\mathcal{E}$, and let B be a finite dimensional subspace of X^*. Then B^*, being a quotient of X, is a quotient of a finite dimensional quotient G of X such that $d_{BM}(G, E) < \lambda$ for some $E \in \mathcal{E}$. Therefore, $B = B^{**}$ is a subspace of $G^* \subseteq X^*$and $d_{BM}(G^*, E^*) < \lambda$, so that X^* is λ-locally $\mathcal{E}^*$. Conversely, suppose that X^* is λ-locally $\mathcal{E}^*$ and let G be a finite dimensional quotient of X, then G^* is a subspace of X^*, and so it is contained in a finite dimensional subspace B of X^* such that $d_{BM}(B, E^*) < \lambda$ for some $E \in \mathcal{E}$. Hence $G \equiv G^{**}$ is a quotient of B^* which is a quotient of X, that is X is λ-colocally $\mathcal{E}$.

(ii) Suppose that X^{**} *is λ-locally* $\mathcal{E} = \mathcal{E}^{**}$, then X^* *is λ-colocally* $\mathcal{E}^*$ by (i). Let A be a finite dimensional subspace of X, then A^* is a quotient of a finite dimensional quotient G of X^* such that $d_{BM}(G, E^*) < \lambda$ for some $E^* \in \mathcal{E}^*$. If $c > \lambda$ is given, then by the principle of local reflexivity there is an isomorphism $T : G^* \to X$ such that $Tx = x$ for all $x \in G^* \cap X$ and $\|T\|\|T^{-1}\| \leqslant c/\lambda$, Johnson et al. [9]. Therefore, $A \hookrightarrow T(G^*) \hookrightarrow X$ and $d_{BM}(T(G^*), E) < c$, since $A = A^{**} \hookrightarrow G^* \hookrightarrow X^{**}$, proving that X *is c-locally* $\mathcal{E}$. Using the same argument, we prove the respective result. □

From Theorem 1.1 and the fact that a Banach space X is an $\mathcal{L}_p$-space, $1 \leqslant p \leqslant \infty$, if and only if X^* is an $\mathcal{L}_q$-space, where q is the conjugate of p, Lindenstrauss and Rosenthal [12, Theorem III(a)], we have the following:

Corollary 1.2. *A Banach space X is an $\mathcal{L}_p$-space, $1 \leqslant p \leqslant \infty$, if and only if it is colocally $\{\ell_p^n\}_{n=1}^{\infty}$.*

Lemma 1.3. *Let X be a Banach space and let A be an n-dimensional subspace of X with Auerbach basis $\{a_i\colon 1 \leqslant i \leqslant n\}$. If for some $0 < \varepsilon < \frac{1}{2n}$ there is a finite dimensional subspace B of X such that for every $i \in \{1, \dots, n\}$, there is an element $b_i \in B$ with $\|a_i - b_i\| < \varepsilon$, then there is an isomorphism $T : X \to X$ such that $T(A) \subseteq B$ and $\|I - T\| \leqslant n\varepsilon$. In particular, for any subspace C of X containing B, there is a subspace D of X containing A such that $d_{BM}(C, D) < \frac{1+n\varepsilon}{1-n\varepsilon}$.*

Proof. Let $\{a_i^*\colon 1 \leqslant i \leqslant n\}$ be the dual basis of $\{a_i\colon 1 \leqslant i \leqslant n\}$ and define $T : X \to X$ by

$$Tx = x + \sum_{i=1}^{n} a_i^*(x)(b_i - a_i), \quad x \in X.$$

Then it is easy to see that T is linear and

$$\|Tx - x\| \leqslant n\varepsilon\|x\|, \qquad (1 - n\varepsilon)\|x\| \leqslant \|Tx\|.$$

Therefore, T is an isomorphism into X with $\|I - T\| \leqslant n\varepsilon$. Since $Ta_i = b_i$, $T(A) \subseteq B$. Now if B is contained in a subspace C of X, taking $D = T^{-1}(C)$, we see that $A \subseteq D$, and the conclusion follows. □

Recall that a family $\mathcal{E} = \{E_\alpha\}_{\alpha \in \Lambda}$ of finite dimensional Banach spaces is said to be **ordered** if $E_\alpha \subseteq E_\beta$,for every $\alpha$and β in the totally ordered set Λ such that $\alpha \leqslant \beta$.

Lemma 1.4. *Let X be a Banach space and let $E = \{E_\alpha\}_\alpha$ be an ordered family of finite dimensional subspaces of X such that $\bigcup E_\alpha$ is dense in X. Then X is $(1 + \epsilon)$-locally E for every $\epsilon > 0$.*

Proof. Let $\epsilon > 0$ and let A be an n-dimensional subspace of X and choose $\epsilon_A = \min\{\epsilon, \frac{1}{2n}\}$. Let $\{a_1, a_2, \dots, a_n\}$ be an $\frac{\delta}{2}$-net of the unit sphere S_A of A, where $0 < \delta < \frac{\epsilon_A}{(2+\epsilon_A)n}$. Now there exist elements $b_1, b_2, \dots, b_n$ in $\bigcup E_\alpha$ such that $\|b_i - a_i\| < \frac{\delta}{2}$, since $\bigcup E_\alpha$ is dense in X. Let $B = \operatorname{span}\{b_1, b_2, \dots, b_n\}$, then there is some $E \in \mathcal{E}$ that contains B since $\mathcal{E}$ is an ordered family of finite dimensional subspaces of X. It follows that for every $a \in A$ such that $\|a\| = 1$ there is $i \in \{1, 2, \dots, n\}$ such that $\|a - b_i\| < \delta$. Hence by Lemma 1.3 there is a subspace D of X, $D \supseteq A$ such that $d_{BM}(E, D) < \frac{1+n\delta}{1-n\delta} < 1 + \epsilon_A < 1 + \epsilon$, and this proves the result. □

A Schauder basis $\{x_i\}_{i=1}^{\infty}$ of a Banach space X is said to be **shrinking** if the sequence $\{x_i^*\}_{i=1}^{\infty}$ of the biorthogonal functionals of $\{x_i\}_{i=1}^{\infty}$ is a basis of X^*, Singer [14].

Theorem 1.5. *If $\{x_i\}_{i=1}^{\infty}$ is a shrinking Schauder basis of a Banach space X and $X_n = \operatorname{span}\{x_1, x_2, \dots, x_n\}$, then X is locally and colocally the family $\{X_n\}$.*

Proof. By Lemma 1.4, X is locally $\{X_n\}$. Since $\{x_i^*\}_{i=1}^{\infty}$ is a basis of X^* then for any $\epsilon > 0$, X^* is $(1+\epsilon)$-locally $\{Y_n\}_n$, where $Y_n = \operatorname{span}\{x_1^*, x_2^*, \dots, x_n^*\}$. It is clear that $X_n^* = \operatorname{span}\{z_1, z_2, \dots, z_n\}$, where z_i is the restriction of x_i^* to X_n. So, to complete the proof it is enough to show that there is a constant c such that for every n the restriction map $\Psi_n : Y_n \to X_n^*$ given by $\Psi_n(x_i^*) = z_i$, is a c-isomorphism. Let $x = \sum_{i=1}^{\infty} a_i x_i$ be an element in X and let $f = \sum_{i=1}^{n} b_i x_i^* \in Y_n$ then $f(x) = f(b_x)$ where $b_x = \sum_{i=1}^{n} a_i x_i$. If c is the basis constant, that is $\| \sum_{i=1}^{n} a_i x_i \| \leqslant c \| \sum_{i=1}^{\infty} a_i x_i \|$ for any $n \in \mathbb{N}$, then

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|f(x)\|}{\|x\|} \leqslant c \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|f(b_x)\|}{\|b_x\|} = c \sup_{\substack{x \in X_n \\ x \neq 0}} \frac{\|f(x)\|}{\|x\|} = c \big\| \Psi_n(f) \big\|,$$

so $\|\Psi_n^{-1}\| \leqslant c$. Since $\|\Psi_n\| \leqslant 1$, Ψ_n is a c-isomorphism. Hence X^* is locally $\{X_n^*\}$ and thus the conclusion follows by Theorem 1.1. □

Our next result is the analogue of [2, Theorem 2], in which we prove that nontrivial twisted sums can be derived with no restrictions imposed on the quotient space of the resulting twisted sum. But first we have the following:

Lemma 1.6. *Let Y, W be Banach spaces and let $\mathcal{E}$ be a family of finite dimensional Banach spaces such that W contains $\mathcal{E}$ uniformly complemented. If* $\operatorname{Ext}(Y, W) = 0$*, then there is a constant $c \geqslant 1$ such that for every $E \in \mathcal{E}$ and every z-linear map $F : Y \to E$ there is a linear map $L : Y \to E$ with* $\operatorname{dist}(F, L) \leqslant cZ(F)$.

Proof. Let $E \in \mathcal{E}$ and let $F : Y \to E$ be a z-linear map. Since W contains $\mathcal{E}$ uniformly complemented there is a k-complemented subspace B of W and a k-isomorphism T of E onto B, and hence there is a linear map $\tilde{L} : Y \to W$ and a constant t such that $\operatorname{dist}(TF, \tilde{L}) \leqslant tZ(TF)$ since the composition map $Y \xrightarrow{F} E \xrightarrow{T} B \hookrightarrow W$is obviously z-linear and $\operatorname{Ext}(Y, W) = 0$, Kalton [10, Proposition 3.3(iii)]. If p_B is a k-projection of W onto B, then for every $y \in Y$ we have

$$\begin{aligned}
\big\|F(y) - T^{-1} p_B \tilde{L}(y)\big\| &= \big\|T^{-1} T F(y) - T^{-1} p_B \tilde{L}(y)\big\| \\
&\leqslant \big\|T^{-1}\big\| \big\|T F(y) - p_B \tilde{L}(y)\big\| \\
&= k \big\|p_B T F(y) - p_B \tilde{L}(y)\big\| \\
&\leqslant k \|p_B\| \big\|T F(y) - \tilde{L}(y)\big\| \\
&\leqslant k^2 t Z(TF) \\
&\leqslant k^2 \|T\| t Z(F) \\
&\leqslant k^3 t Z(F),
\end{aligned}$$

which implies that $\operatorname{dist}(F, T^{-1} p_B \tilde{L}) \leqslant k^3 t Z(F)$. The proof is completed by putting $L = T^{-1} p_B \tilde{L}$ and $c = k^3 t$. □

Theorem 1.7. *Let $\mathcal{E}$ be a family of finite dimensional Banach spaces and let W be a Banach space containing $\mathcal{E}$* uniformly complemented. *If Y is a Banach space such that* $\operatorname{Ext}(Y, W) = 0$, *then* $\operatorname{Ext}(Y, Z) = 0$ *for every Banach space Z complemented in its bidual and colocally $\mathcal{E}$.*

Proof. Let Z be a Banach space complemented in its bidual and λ-colocally $\mathcal{E}$ for some $\lambda > 1$, and let $F : Y \to Z$ be a z-linear map. We shall show that F is trivial by constructing a linear map $L : Y \to Z^{**}$ such that $\operatorname{dist}(F, L) < \infty$. So, let $\mathcal{C}$ be the net of all finite codimensional subspaces

A of Z directed by reverse inclusion, then for every $A \in \mathcal{C}$, there is $B_A \in \mathcal{C}$ and $E_A \in \mathcal{E}$ such that Z/A is a quotient of Z/B_A and $d_{BM}(Z/B_A, E_A) < \lambda$. Let $T_A : Z/B_A \to E_A$ be a λ-isomorphism and let $q_A : Z \to Z/B_A$ be the canonical quotient map. Hence the composition map $Y \xrightarrow{F} Z \xrightarrow{q_A} Z/B_A \xrightarrow{T_A} E_A$ is obviously a z-linear map. Since $\mathrm{Ext}(Y, W) = 0$ then there is a constant $c \geqslant 1$ (does not depend on E_A) and a linear map $L_A : Y \to E_A$ such that $\mathrm{dist}(T_A q_A F, L_A) \leqslant cZ(F)$ by Lemma 1.6. Therefore

$$\left\| q_A F(y) - T_A^{-1} L_A(y) \right\| \leqslant c\lambda Z(F)\|y\|, \quad y \in Y.$$

Let $\mathcal{U}$ be an ultrafilter which refines the corresponding order filter on $\mathcal{C}$ and note that $\{A^{\perp}: A \in \mathcal{C}\}$ is the net of all finite dimensional subspaces of Z^*, and $A^{\perp} \subseteq B^{\perp}$ when $B \subseteq A$, $A, B \in \mathcal{C}$. By Diestel [6, 8.8] and Rudin [13, 4.7, 4.8], there is a canonical isometric embedding

$$J : Z^* \to \left(\prod_{A \in \mathcal{C}} A^{\perp}\right)_{\mathcal{U}} \subseteq \left(\prod_{A \in \mathcal{C}} B_A^{\perp}\right)_{\mathcal{U}} \equiv \left(\prod_{A \in \mathcal{C}} (Z/B_A)^*\right)_{\mathcal{U}}$$

given by $J(f) = (f_A)_{\mathcal{U}}$, $f \in Z^*$, where $f_A = f$ if $f \in B^{\perp}$ and $f_A = 0$ otherwise. Therefore, by setting

$$(f_A)_{\mathcal{U}}\big((z + B_A)_{\mathcal{U}}\big) = \lim_{\mathcal{U}} \big(f_A(z + B_A)\big),$$

$(\prod_{A \in \mathcal{C}} (Z/B_A)^*)_{\mathcal{U}}$ embeds isometrically into $(\prod_{A \in \mathcal{C}} (Z/B_A))_{\mathcal{U}}^*$ (see Diestel [6, 8.3]), where the norm satisfies $\|(f_A)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|f_A\|$ and so we have

$$J : Z^* \to \left(\prod_{A \in \mathcal{C}} B_A^{\perp}\right)_{\mathcal{U}} \equiv \left(\prod_{A \in \mathcal{C}} (Z/B_A)^*\right)_{\mathcal{U}} \hookrightarrow \left(\prod_{A \in \mathcal{C}} (Z/B_A)\right)_{\mathcal{U}}^*.$$

If $Q : (\prod_{A \in \mathcal{C}} (Z/B_A))_{\mathcal{U}} \to Z^{**}$ is the restriction of the adjoint operator

$$J^* : \left(\prod_{A \in \mathcal{C}} (Z/B_A)\right)_{\mathcal{U}}^{**} \to Z^{**}$$

then

$$\big(Q\big((z + B_A)_{\mathcal{U}}\big)\big)(f) = \big(J(f)\big)\big((z + B_A)_{\mathcal{U}}\big) = \lim_{\mathcal{U}} f_A(z + B_A) = \lim_{\mathcal{U}} \big(f_A(z)\big),$$

for every $f \in Z^*$ and $(z + B)_{\mathcal{U}} \in (\prod_{A \in \mathcal{C}_Z} (Z/B_A))_{\mathcal{U}}$. Thus, if $f \in Z^*$ and $y \in Y$,

$$\big(Q\big((q_A F(y))_{\mathcal{U}}\big)\big)(f) = \lim_{\mathcal{U}} \big(f_A\big(F(y)\big)\big) = f\big(F(y)\big) = F(y)(f).$$

That is, $Q((q_A F(y))_{\mathcal{U}}) = F(y)$ for every $y \in Y$. Now define $L : Y \to Z^{**}$ by $L(y) = Q((T_A^{-1} L_A(y))_{\mathcal{U}})$, and let $\pi : Z^{**} \to Z$ be a bounded projection of Z^{**} onto Z, then $\pi \circ L$ is a linear map from Y into Z with

$$\begin{aligned}
\left\| F(y) - L(y) \right\| &= \left\| Q\big((q_A F(y))_{\mathcal{U}}\big) - Q\big(T_A^{-1} L_A(y)\big)_{\mathcal{U}} \right\| \\
&\leqslant \|Q\| \lim_{\mathcal{U}} \left\| q_A F(y) - T_A^{-1} L_A(y) \right\| \\
&\leqslant c\lambda Z(F)\|Q\|\|y\|.
\end{aligned}$$

Hence, $\mathrm{dist}(F, \pi L) < \infty$. That is, F is trivial, by Kalton and Peck [11, Theorem 2.5], and the theorem is proved. □

Note that if the bidual X^{**} of a Banach space X has a shrinking basis $\{x_i\}_{i=1}^{\infty}$, then it is colocally $\{X_n\}$, by Theorem 1.5, where $X_n = \text{span}\{x_1, x_2, \dots, x_n\}$. Therefore, if Y is a Banach space such that $\text{Ext}(Y, X) = 0$, then $\text{Ext}(Y, X^{**}) = 0$, by Theorem 1.7.

The next result follows directly from Theorem 1.7.

Corollary 1.8. *Let X be a Banach space, then* $\text{Ext}(X, \ell_1) = 0$ *if and only if* $\text{Ext}(X, L_1(\mu) = 0$ *for every measure μ.*

It is important to note that $\text{Ext}(X, \ell_1) = 0$ does not imply that $\text{Ext}(X, \mathcal{L}_1) = 0$ for every $\mathcal{L}_1$-space as the canonical projective presentation $0 \to Y \to \ell_1 \to L_1(0,1) \to 0$ shows.

The proof of the following known fact is immediate by Theorem 1.7.

Corollary 1.9. *If a Banach space Z is locally $\{\ell_\infty^n\}_n$ and complemented in its bidual, then* $\text{Ext}(Y, Z) = 0$ *for any Banach space Y.*

Proof. Since $\text{Ext}(Y, \ell_\infty) = 0$ for any Banach space Y, Diestel [5, Chapter VII, Theorem 3], then $\text{Ext}(Y, Z) = 0$ for any Banach space Z complemented in its bidual and colocally $\{\ell_\infty^n\}_n$, by Theorem 1.7, that is locally $\{\ell_\infty^n\}_n$ by Corollary 1.2. □

Lemma 1.10. *Let Y and Z be Banach spaces, then $BL(Y, Z^*) = BL(Z, Y^*)$.*

Proof. Define $\Psi : BL(Y, Z^*) \to BL(Z, Y^*)$ by $\Psi(T) = T^* \circ i$, where $T \in BL(Y, Z^*)$, $i : Z \to Z^{**}$ is the inclusion map and $T^* \in BL(Z^{**}, Y^*)$ is the adjoint operator of T. Clearly Ψ is a bounded linear map. To show that Ψ is an isometry, let $T \in BL(Y, Z^*)$, and let g be an element in the unit ball $(Z^{**})_1$ of Z^{**}. Then $g = w^*\text{-}\lim z_n$, for some sequence (z_n) in $(Z)_1$ since $(\dot{Z})_1$ is w^*-dense in $(Z^{**})_1$. Now T^* is w^*-continuous hence

$$\left\|T^*(g)\right\| = \left\|w^*\text{-}\lim T^* \circ i(z_n)\right\| \leqslant \left\|T^* \circ i\right\| \lim \|z_n\| = \left\|T^* \circ i\right\| \|g\|.$$

Therefore $\|T\| = \|T^*\| \leqslant \|T^* \circ i\| \leqslant \|T^*\|$, proving that Ψ is an isometry. □

Theorem 1.11. *Let Y and Z be Banach spaces, then the vector spaces* $\text{Ext}(Z, Y^*)$ *and* $\text{Ext}(Y, Z^*)$ *are isomorphic.*

Proof. Let $0 \to K \xrightarrow{i} P \xrightarrow{q} Y \to 0$ be a projective presentation of Y, then its dual sequence $0 \to Y^* \to P^* \to K^* \to 0$ is an injective presentation of Y^* (cf. Castello and González [4, Lemma 2.2.d]). Therefore, P and P^* induce the following exact sequences of vector spaces:

$$0 \to BL(Z, Y^*) \to BL(Z, P^*) \to BL(Z, K^*) \to \text{Ext}(Y, Z^*) \to 0,$$
$$0 \to BL(Z, Y^*) \to BL(Z, P^*) \to BL(Z, K^*) \to \text{Ext}(Z, Y^*) \to 0$$

via push-out and pull-out, respectively [2, p. 8], which are equivalent by Lemma 1.10. Hence, $\text{Ext}(Y, Z^*)$ and $\text{Ext}(Z, Y^*)$ are isomorphic. □

Corollary 1.12. *Let Y and Z be two Banach spaces. Then* $\text{Ext}(Y, Z^*) = 0$ *if and only if* $\text{Ext}(Z, Y^*) = 0$.

Remark 1.13 *(Cabello and Castillo [1, Theorem 3]).* Let $F : Z \to Y$ be a z-linear map. Then there is a homogeneous map $H : Y^* \to Z'$ that satisfies

$$\|H(y^*) - y^* \circ F\| \leqslant Z(F)\|y^*\|,$$

for all $y^* \in Y^*$. Given a Hamel basis (f_α) of Y^*, a linear map $L_H : Y^* \to Z'$ can be defined by $L_H(f_\alpha) = H(f_\alpha)$. The dual map F^* of F is the map $L_H - H$, it is a z-linear map, $F^*(y^*) \in Z^*$ for every $y^* \in Y^*$ and $Z(F^*) \leqslant 2Z(F)$. The map F^* is unique up to equivalence.

Recall that two families $\mathcal{E}$ and $\mathcal{F}$ of finite dimensional Banach spaces are said to satisfy $\mathrm{Ext}(\mathcal{F}, \mathcal{E}) = 0$ **uniformly** if there is a constant c such that, for every couple of spaces $A \in \mathcal{E}$ and $B \in \mathcal{F}$ and every z-linear map $F : B \to A$, there is a linear map $L : B \to A$ such that $\mathrm{dist}(F, L) \leqslant cZ(F)$, Cabello and Castillo [2, p. 7].

Lemma 1.14. $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ *uniformly if and only if* $\mathrm{Ext}(\mathcal{F}^*, \mathcal{E}^*) = 0$ *uniformly.*

Proof. Suppose that $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ uniformly and let $A \in \mathcal{E}$, $B \in \mathcal{F}$, $F : B^* \to A^*$ be a z-linear map. The dual z-linear map $F \equiv F^{**} : B^* \equiv (B^*)^{**} \to (A^*)^{**} \equiv A^*$ can be written as $F = F^{**} = L_H - H$ for some homogeneous map $H : B^* \to A'$ that satisfies $\|b^* \circ F^* - H(b^*)\| \leqslant Z(F^*)\|b^*\|$ for all $b^* \in B^*$ and a linear map $L_H : B^* \to A^*$, Cabello and Castillo [1, Theorem 3 and Remark 1]. Since $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ uniformly, there is a linear map $T : A \to B$ and a constant c depends on $\mathcal{E}$ and $\mathcal{F}$ such that $\|F^*(a) - T(a)\| \leqslant cZ(F^*)\|a\|$ for all $a \in A$. The dual map $T^* : B^* \to A^*$ satisfies $T^*(b^*)(a) = b^*(T(a))$ for all $a \in A$. Therefore,

$$\begin{aligned}\|F(b^*) - L_H(b^*) + T^*(b^*)\| &= \|-H(b^*) + T^*(b^*) \pm b^* \circ F^*\| \\ &\leqslant \|b^* \circ F^* - H(b^*)\| + \|T^*(b^*) - b^* \circ F^*\| \\ &\leqslant Z(F^*)\|b^*\| + \|b^*\|\|T - F^*\| \\ &\leqslant (c+1)Z(F)\|b^*\|,\end{aligned}$$

for all $b^* \in B^*$ since $Z(F) \leqslant Z(F^*)$. Hence $\mathrm{Ext}(\mathcal{F}^*, \mathcal{E}^*) = 0$ uniformly. The converse is obvious since $\mathcal{E} = \mathcal{E}^{**}$. □

It is shown that if $\mathcal{E}$ and $\mathcal{F}$ are families of finite dimensional Banach spaces such that $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ uniformly, and if Y and Z are Banach spaces such that Y is locally $\mathcal{E}$, Z is locally $\mathcal{F}$ and Z is complemented in its bidual, then $\mathrm{Ext}(Y, Z) = 0$, Cabello and Castillo [2, Theorem 3]. Similar results involving colocality families are given in the following:

Theorem 1.15. *If Y and Z are Banach spaces such that Y is colocally $\mathcal{E}$, Z is colocally $\mathcal{F}$ complemented in its bidual, and* $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ *uniformly, then* $\mathrm{Ext}(Y, Z) = 0$.

Proof. It is immediate by Theorem 1.1, Lemma 1.14 and [2, Theorem 3]. □

Theorem 1.16. *Let Y, Z be Banach spaces and let $\mathcal{E}$, $\mathcal{F}$ be two families of finite dimensional Banach spaces such that Y is locally $\mathcal{E}$, Z is colocally $\mathcal{F}$ and is complemented in its bidual. If* $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ *uniformly, then* $\mathrm{Ext}(Y, Z) = 0$.

Proof. Suppose that Y is λ-locally $\mathcal{E}$, Z is $\acute{\lambda}$-colocally $\mathcal{F}$ and let $F : Y \to Z$ be a z-linear map. Then there is a cofinal subnet $\mathcal{G}$ of the net of all finite dimensional subspaces of Y such that for every $G \in \mathcal{G}$ there is $E \in \mathcal{E}$ with $d_{BM}(G, E) \leqslant \lambda$. Let $\mathcal{C}$ be the net of all finite codimensional subspaces of Z directed by reverse inclusion, then for each $A \in \mathcal{C}$, let B_A and q_A be as described in the proof of Theorem 1.7.

For each $G \in \mathcal{G}$, let F_G be the restriction of F to G, then clearly $Z(F_G) \leqslant Z(F)$ and the composition map $q_A F_G : G \to Z/B_A$ is a trivial z-linear map. Since $\mathrm{Ext}(\mathcal{E}, \mathcal{F}) = 0$ uniformly, there is a constant c and a linear map $L_G : G \to Z/B_A$ such that for all $y \in G$,

$$\left\| q_A F_G(y) - L_G(y) \right\| \leqslant c Z(F_G) \|y\| \leqslant c\lambda\acute{\lambda} Z(F) \|y\|.$$

Note that

$$\left\| L_G(y) \right\| \leqslant \left\| q_A F_G(y_G) - L_G(y) \right\| + \left\| q_A F_G(y) \right\| \leqslant c\lambda\acute{\lambda} Z(F) \|y\| + \left\| F(y) \right\|.$$

Let $\mathcal{V}$ be an ultrafilter refining the order filter on $\mathcal{G}$ and define a map $L_A : Y \to Z/B_A$ by

$$L_A(y) = w^*\text{-}\lim_{\mathcal{V}} L_G(y_G),$$

where $y_G = y$ if $y \in G$, and 0 otherwise. It is easy to see that L_A is a linear map and well defined since for all $G \in \mathcal{G}$. Hence,

$$\left\| q_A F(y) - L_A(y) \right\| \leqslant c\lambda\acute{\lambda} Z(F) \|y\|,$$

which implies that $q_A F$ is trivial. As in the proof of the Theorem 1.7, it is possible to find a linear map $L : Y \to Z$ such that $\|F(y) - L(y)\| \leqslant k\|y\|$for some constant k, proving that $\mathrm{Ext}(Y, Z) = 0$. □

2. Some applications

The **Schreier space** S is the completion of the space of finite sequences with respect to the following norm:

$$\|x\| = \sup_A \left(\sum_{j \in A} |x_j| \right),$$

where the supremum is taken over all "admissible" subsets of $\mathbb{N}$, which are defined as the finite subsets $A = \{n_1, n_2, \ldots, n_k\}$ of $\mathbb{N}$ such that $n_1 < n_2 < \cdots < n_k$ and $k \leqslant n_1$, Castello and González [4, p. 119].

Remark 2.1. Note that the Schreier space S contains $\{l_\infty^n\}$ uniformly complemented, since it contains c_0, Castillo and González [3, p. 167]. Also, S contains $\{\ell_1^n\}$ uniformly complemented, since for each $n \in \mathbb{N}$, the 1-complemented subspace $F_n = \{(x_i) \in S\colon \operatorname{supp}(x_i) \subseteq \{n, n+1, \ldots, 2n-1\}\}$ of S is isometrically isomorphic to ℓ_1^n.

Example 2.2. Recall that ℓ_1 is locally and colocally $\{\ell_1^n\}$ and note that c_0 is locally $\{l_\infty^n\}$.

(i) Since $\mathrm{Ext}(c_0, \ell_1) \neq 0$ and $\mathrm{Ext}(\ell_2, \ell_1) \neq 0$ [2, Examples 4.1 and 4.3], we have $\mathrm{Ext}(c_0, S) \neq 0$ and $\mathrm{Ext}(l_2, S) \neq 0$, by Remark 2.1 and Theorem 1.7.

(ii) Since $\mathrm{Ext}(c_0, \ell_1) \neq 0$, and since S and S^{**} contain $\{l_\infty^n\}_n$ uniformly complemented, we have $\mathrm{Ext}(S, \ell_1) \neq 0$ and $\mathrm{Ext}(S^{**}, \ell_1) \neq 0$ [2, Theorem 2]. Hence, $\mathrm{Ext}(S, S) \neq 0$, $\mathrm{Ext}(S, S^{**}) \neq 0$, $\mathrm{Ext}(S^{**}, S) \neq 0$, and $\mathrm{Ext}(S^{**}, S^{**}) \neq 0$, by Theorem 1.7.

(iii) Since $\mathrm{Ext}(S, S^{**}) \neq 0$, we have $\mathrm{Ext}(S^*, S^*) \neq 0$, by Corollary 1.12.

The **Johnson–Lindenstrauss space JL** is defined to be the completion of the linear span of $c_0 \cup \{\chi_i\colon i \in I\}$ in ℓ_∞ with respect to the norm:

$$\left\| y = x + \sum_{j=1}^{k} a_{i(j)} \chi_{i(j)} \right\| = \max\left\{ \|y\|_\infty, \left\| (a_i)_{i\in I} \right\|_{\ell_2(I)} \right\}, \quad x \in c_0,\ a_{i(j)} \text{ are scalars,}$$

where χ_i is the characteristic function of A_i, $\{A_i\}_{i\in I}$ is an almost disjoint uncountable family of infinite subsets of $\mathbb{N}$.

Example 2.3. Since JL/c_0 is isomorphic to $\ell_2(I)$ [4, p. 126], and since ℓ_1 is projective, the dual sequence $0 \to \ell_2(I) \to JL^* \to \ell_1 \to 0$ of the exact sequence $0 \to c_0 \to JL \to \ell_2(I) \to 0$ splits. That is, $JL^* = \ell_1 \oplus \ell_2(I)$. Hence, $\mathrm{Ext}(c_0, JL^*) \neq 0$, since $\mathrm{Ext}(c_0, \ell_1) \neq 0$ [2, Examples 4.3], and so $\mathrm{Ext}(JL, \ell_1) \neq 0$, by Corollary 1.12. Therefore, $\mathrm{Ext}(JL, S) \neq 0$ and $\mathrm{Ext}(JL, S^{**}) \neq 0$ by Theorem 1.7.

The **James space** $(J, \|.\|)$ is the Banach space of all real sequences $x = (a_1, a_2, \ldots)$ such that $\lim_{n\to\infty} a_n = 0$ and $\sup(\sum_{i=1}^{n} (a_{p_{2i-1}} - a_{p_{2i}})^2) < \infty$, where the supremum is taken over all choices of n and of positive integers $p_1 < p_2 < \cdots < p_{2n}$, equipped with the norm

$$\|x\| = \sup\left(\sum_{i=1}^{n} (a_{p_{2i-1}} - a_{p_{2i}})^2 \right)^{1/2}.$$

The unit vectors $\{e_n\}_{n=1}^{\infty}$ form a basis of J, Fetter and Gamboa de Buen [7, p. 12].

The **James Tree space** $(JT, \|.\|)$ is defined to be the completion of the space of finite sequences over the dyadic tree Δ with respect to the norm

$$\|x\| = \sup_{n\in N} \sup_{S_1,\ldots,S_n} \left[\sum_{i=1}^{n} \left(\sum_{\alpha\in S_i} x_\alpha \right)^2 \right]^{1/2},$$

where the supremum is taken over all finite sets of pair wise disjoint segments of Δ. The space JT is an example of a separable Banach space that does not contain ℓ_1, Castillo and Gonzalez [4, p. 133].

Lemma 2.4. *The predual B of the James tree space JT contains $\{\ell_1^n\}_{n=1}^{\infty}$ uniformly complemented.*

Proof. Since c_0 is finitely represented in the James space J and JT contains J [7, 2.b.8 and 3.a.7], then JT contains $\{\ell_\infty^n\}_{n=1}^{\infty}$ uniformly which implies that it contains $\{\ell_\infty^n\}_{n=1}^{\infty}$ uniformly complemented. Choose $c > 1$ and let G_n be a subspace of $JT = B^*$ such that G_n is c-isomorphic to ℓ_∞^n, $n \in \mathbb{N}$. Since G_n is w^*-closed in B^*, there is a subspace D_n of B such that $G_n = (B/D_n)^*$ [13, 4.8]. Therefore ℓ_1^n is c-isomorphic to B/D_n. Now let ψ_n be a c-isomorphism of B/D_n onto ℓ_1^n and let $q_n : B \to B/D_n$ be the quotient map, then $\psi_n q_n(cB_B(0,t)) \supseteq B_{\ell_1}$, for any prefixed $t > 1$. Let $\{e_i^n\colon i = 1, 2, \ldots, n\}$ be the standard basis of ℓ_1^n, then there is $x_i^n \in cB_B(0,t)$ such that $\psi_n q_n(x_i^n) = e_i^n$. It is clear that $x_1^n, x_2^n, \ldots, x_n^n$ are linearly independent. Let B_n be the subspace of B generated by $\{x_1^n, x_2^n, \ldots, x_n^n\}$, define $T_n : \ell_1^n \to B$ by $T_n(\sum_{i=1}^{n} \lambda_i e_i^n) = \sum_{i=1}^{n} \lambda_i x_i^n$. Thus

$$\left\| T_n\left(\sum_{i=1}^{n} \lambda_i e_i^n \right) \right\| = \left\| \sum_{i=1}^{n} \lambda_i x_i^n \right\| \leqslant ct \sum_{i=1}^{n} |\lambda_i| = ct \left\| \sum_{i=1}^{n} \lambda_i e_i^n \right\|_1,$$

from which we see that $\|T_n\| \leqslant ct$.

On the other hand, $T_n^{-1}: B_n \to \ell_1^n$ satisfies

$$\left\| T_n^{-1}\left(\sum_{i=1}^n \lambda_i x_i^n\right)\right\| = \left\| \sum_{i=1}^n \lambda_i e_i^n \right\| = \left\| \psi_n q_n\left(\sum_{i=1}^n \lambda_i x_i^n\right)\right\| \leqslant \|\psi_n\| \|q_n\| \left\| \sum_{i=1}^n \lambda_i x_i^n \right\|$$
$$\leqslant c\left\| \sum_{i=1}^n \lambda_i x_i^n \right\|,$$

which implies that $\|T_n^{-1}\| \leqslant c$. Hence ℓ_1^n is ct-isomorphic to B_n. Now consider the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{q_n} & B/D_n \\ {\scriptstyle T_n}\uparrow & & \downarrow{\scriptstyle \psi_n} \\ \ell_1^n & \xrightarrow{id_n} & \ell_1^n \end{array}$$

It is easily seen that $\psi_n q_n T_n = id_n$, so $T_n \psi_n q_n : B \to T_n(\ell_1^n)$ is a projection with norm $\leqslant c^2 t$. Therefore the result holds. $\square$

Example 2.5. Consider the James tree space JT and its predual B. By Lemma 2.4, B contains $\{\ell_1^n\}_{n=1}^\infty$ uniformly complemented, and hence JT^* contains $\{\ell_1^n\}_{n=1}^\infty$ uniformly complemented.

(i) Since $\mathrm{Ext}(c_0, \ell_1) \neq 0$ and $\mathrm{Ext}(\ell_2, \ell_1) \neq 0$ [2, Examples 4.1 and 4.3], we have $\mathrm{Ext}(c_0, B) \neq 0$, $\mathrm{Ext}(c_0, JT^*) \neq 0$, $\mathrm{Ext}(\ell_2, B) \neq 0$ and $\mathrm{Ext}(\ell_2, JT^*) \neq 0$, by Theorem 1.7.

(ii) Since $\mathrm{Ext}(S, \ell_1) \neq 0$, by Example 2.2, we have $\mathrm{Ext}(S, B) \neq 0$ and $\mathrm{Ext}(S, JT^*) \neq 0$, by Theorem 1.7.

(iii) Since $\mathrm{Ext}(JL, \ell_1) \neq 0$, by Example 2.3, we have $\mathrm{Ext}(JL, B) \neq 0$ and $\mathrm{Ext}(JL, JT^*) \neq 0$, by Theorem 1.7.

Example 2.6. The Argyros–Deliyanni space AD and Tsirelson's space T, being asymptotically ℓ_1, contain $\{\ell_1^n\}$ uniformly complemented. Since $\mathrm{Ext}(c_0, \ell_1) \neq 0$, we have $\mathrm{Ext}(c_0, AD) \neq 0$ and $\mathrm{Ext}(c_0, T) \neq 0$, by Theorem 1.7.

References

[1] F. Cabello Sànchez, J.M.F. Castillo, Duality and twisted sums of Banach spaces, J. Funct. Anal. 175 (1) (2000) 1–16.

[2] F. Cabello Sánchez, J.M. Castillo, Uniform boundedness and twisted sums of Banach spaces, Houston J. Math. 30 (2) (2004) 523–536.

[3] J.M. Castillo, M. González, An approach to Schreier's space, Extracta Math. 6 (1991) 166–169.

[4] J.M.F. Castillo, M. González, Three-Space Problems in Banach Space Theory, Lecture Notes in Math., vol. 1667, Springer, Berlin, 1997.

[5] J. Diestel, Sequences and Series in a Banach Space, Grad. Texts in Math., vol. 92, Springer, New York, 1984.

[6] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, Cambridge Stud. Adv. Math., vol. 43, Cambridge Univ. Press, Cambridge, 1995.

[7] H. Fetter, B. Gamboa de Buen, The James Forest, London Math. Soc. Lecture Note Ser., vol. 236, Cambridge Univ. Press, Cambridge, 1997.

[8] S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew. Math. 313 (1980) 72–104.

[9] W.B. Johnson, H.P. Rosenthal, M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, Israel J. Math. 9 (1971) 488–506.

[10] N.J. Kalton, The three space problem for locally bounded F-spaces, Compos. Math. 37 (1978) 243–276.
[11] N.J. Kalton, N.T. Peck, Twisted sums of sequence spaces and the three space problem, Trans. Amer. Math. Soc. 255 (1979) 1–30.
[12] J. Lindenstrauss, H.P. Rosenthal, The $\mathcal{L}_p$ spaces, Israel J. Math. 7 (1969) 325–349.
[13] W. Rudin, Functional Analysis, second ed., Internat. Ser. in Pure and Appl. Math., McGraw–Hill, New York, 1991.
[14] I. Singer, Bases in Banach Spaces. I, Springer, Berlin, 1970.