KING SAUD UNIVERSITY
Final Exam Of Math **282**, Spring of 1422-1423
Duration Three Hours

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1. Prove or disprove the following :

   (a) Let A be a non empty subset of $\mathbf{R}$ and A is bounded below. Then the set $-A$ is bounded above and $inf\, A = -\, sup(-A)$.

   (b) Let $f : I \to \mathbf{R}$ be differentiable on I. If f is strictly increasing then $f'(x) > 0$.

   (c) For $b > 1$, the sequence $(b^n/n!)$ converges to zero.

   (d) Let (x_n) and (y_n) be sequences of real numbers and suppose $x_n \leq y_n$ for all n.
   If $lim(x_n) = -\infty$ then $lim(y_n) = -\infty$.

   (e) The function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = sin(1/x)$ is uniformly continuous.

2. A- Let $f : I \to \mathbf{R}$ be an increasing function where I is an interval and suppose c is an interior point of I. Show that
   $lim_{x \to c^-} f(x) = sup\{f(x) |\ x \in I \text{ and } x < c\}$.

   B- Let c be an interior point of the interval I. Suppose $f : I \to \mathbf{R}$ has a relative maximum at c, if $f'(c)$ exists then **prove** that $f'(c) = 0$.
   Is the converse true, i.e if $f'(c) = 0$ does this imply that f has a relative extremum there ?

   C- Let $f : R \to R$ be defined by

   $$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q}^c \end{cases}$$

   Show that f is deffrentiable at $x = 0$, and find $f'(0)$.

3. A- Let $f : [a, b] \to \mathbf{R}$ be a continuous function. **Prove** that $f([a, b])$ is bounded.

   B- **State** the preservation of the intervals theorem.

Let $f : [a,b] \to \mathbf{R}$ be a continuous function, is it true that $f([a,b]) = [f(a), f(b)]$, justify.

Does this contradict the preservation of the intervals theorem ? justify.

C- **State** the Darboux's theorem.

Let $f : [0,2] \to \mathbf{R}$ be continuous function on $[0,2]$ and differentiable on $(0,2)$. Suppose f satisfies $f(0) = 0$, and $f(1) = f(2) = 1$.

Show that there exists $c \in (0,2)$ such that $f'(c) = 1/3$.

4. A- **State** and **prove** the Mean value theorem.

B- Use the Mean value theorem to prove

$$(\frac{x-1}{x}) < \log x < x - 1, \quad \text{for } x > 1.$$

C- Use the ϵ and δ definition to show that

$$lim_{n\to\infty}(\frac{1}{n} - \frac{1}{n+1}) = 0.$$

D- Evaluate the following limit

$$lim_{x\to\infty}\frac{x + \log x}{x \log x}, \quad x \in (0,\infty).$$

Good Luck.