KING SAUD UNIVERSITY

Final Exam Of Math 282, Spring of 1422-1423

Duration three hours

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1. Prove or disprove the following statements :

   (a) Every continuous function on an open interval is uniformly continuous.

   (b) If $f : \mathrm{R} \to \mathrm{R}$ satisfies $|f(x) - f(y)| \leq K|x - y|$, for all $x, y \in \mathrm{R}$ and $K > 0$, then f is uniformly continous.

   (c) Let $f : \mathrm{R} \to \mathrm{R}$ be a continuous function, then $A = \{x \in \mathrm{R} | f(x) = 2\}$ is a closed subset of R.

   (d) If c is a cluster point of a set $A \subset \mathrm{R}$, and $lim_{x \to c}(f(x))^2 = L$, then f has a limit at c.

   (e) If (x_n), (y_n) and (z_n) are sequences such that $x_n \leq y_n \leq z_n$ for all n, and $lim(x_n) = lim(z_n) = L$. Then the sequence (y_n) is convergent and $lim(y_n) = L$.

   (f) Let $f : \mathrm{I} \to \mathrm{R}$ be a differentiable function, suppose f is strictly increasing function then $f'(x) > 0$.

   (g) Every continuous function maps an open interval onto an open interval.

2. A- Show that

$$lim_{x \to 0^-} e^{1/x} = 0, \ x \neq 0.$$

   B- State and prove the Bolzano's intermediate value theorem.

   C- Let $f : [0, 1] \to \mathrm{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, 1/2]$ such that $f(c) = f(c + 1/2)$.

3. A- Let $f : \mathrm{I} \to \mathrm{R}$ be an increasing function on an interval I. Suppose that $c \in \mathrm{I}$ is not an end point of I. Show that

$$lim_{x \to c^-} f(x) = sup\{f(x) : x \in \mathrm{I}, x < c\}.$$

   B- Let $\alpha > 1$. Show that

$$(1 + x)^{\alpha} \geq 1 + \alpha x, \quad \text{for all } x > -1.$$

C- Given that the function $f(x) = 4x^3 + 2x + 6$ for $x \in R$ has an inverse f^{-1} on R. Find the value $(f^{-1})'(y)$ at $x = 3$.

4. A- Let $f : [0, 2] \to R$ be continuous on $[0, 2]$ and differentiable on $(0, 2)$, and that $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.

1- Show that there exists $c_1 \in (0, 1)$ such that $f'(c_1) = 1$

2- Show that there exists $c_2 \in (1, 2)$ such that $f'(c_2) = 0$

3- Show that there exists $c \in (0, 2)$ such that $f'(c) = 1/3$.

B- Suppose that f and g are continuous and differentiable on $[b, \infty)$, that $lim_{x\to\infty} f(x) = lim_{x\to\infty} g(x) = 0$, and that $g(x) \neq 0$ and $g'(x) \neq 0$ for $x > b$. Then

$$lim_{x\to\infty} f(x)/g(x) = lim_{x\to\infty} f'(x)/g'(x).$$

C- Evaluate the following limit

$$lim_{x\to\infty} \frac{\log x}{\sqrt{x}}, \quad x \in (0, \infty).$$

Good Luck