King Saud University

Department of Mathematics

fall 1426-1427

Final Exam
Math 282

1. (a) Let $f, g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and $\lim\limits_{x\to c} g(x) = 0$. Prove that $\lim\limits_{x\to c} (fg)(x) = 0$.

   (b) Use the fact in part (a) to show that the function $h(x) = x \sin \frac{1}{x}, x \neq 0$, has a limit at $x = 0$. Then redefine the function h so that the resulting function H is contionuous at $x = 0$.

   (c) Let f and g be defined on (a, ∞) and suppose that $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} g(x) = \infty$. Prove that $\lim\limits_{x\to\infty} f \circ g(x) = L$.

2. (a) Prove that if f is a continuous function on a closed bounded interval $I = [a, b]$, then f is bounded on I.

   (b) Show that the hypothesis in (a) that I is closed is essential.

3. (a) State Darboux's Theorem.

   (b) Use (a) to show that, if $I = [a, b]$ is a closed bounded interval and $f : I \to \mathbb{R}$ is differentiable on I such that $f'(x) \neq 0, \forall x \in I$, then $f'(x) > 0, \forall x \in I$ or $f'(x) < 0, \forall x \in I$.

   (c) Find the following limits (if exist)

   (i) $\lim_{x\to 1^+} \frac{x}{x-1}$ (by definition).

   (ii) $\lim_{x\to 1} 2x + sgn x$.

   (iii) $\lim_{x\to 0^+} \frac{\ln(x+1)}{\sin x}, x \in \left(0, \frac{\pi}{2}\right)$.

   (iv) $\lim_{x\to 0^+} (\sin x)^x, x \in (0, \pi)$.

4. (a) State and prove Roll's Theorem.

   (b) Let $f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{Q}^c. \end{cases}$

   (i) Discuss the continuity of f on $\mathbb{R}$.

   (ii) Is f' differentiable at 0.

5. Prove or disprove

(a) If (x_n) and $(y_n)$are properly divergent sequences, $y_n \neq 0, \forall n \in \mathbb{N}$, then $\left(\frac{x_n}{y_n}\right)$ is properly divergent.

(b) If $K > 0, f : \mathbb{R} \to \mathbb{R}$ satisfies

$$| f(x) - f(y) | \leq K | x - y |, \forall x, y \in \mathbb{R},$$

then f is continuous on $\mathbb{R}$.

(c) If f is strictly decreasing on an interval I, then $f'(x) < 0, \forall x \in I$.

(d) If S is a bounded set in $\mathbb{R}$ and S_0 is a nonempty subset of S, then $\inf S \leq \inf S_0$.

(e) $\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$.

Bonus:

Prove or disprove that if $f : A \to R$ is differentiable at $c \in A$, then f' is continuous at c.

Good Luck.

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