KING SAUD UNIVERSITY
MATH 482
Problem sheet 1

Dr. Fatima Azmi

---

1. 1) Describe all subspaces of $\mathbf{R^2}$ and $\mathbf{R^3}$.

   2) Let $W = \{A \in M_{3\times3}(\mathbf{R}) \ni A^t = A\}$ Is W a subspace of $M_{3\times3}(\mathbf{R})$.

2. Let V and W be normed vector spaces of dimension n and m respectively. Let $T : V \to W$ be a linear transformation. Show that $Ker(T)$ is a subspace of V and $Im((T)$ is a subspace of W.

3. Determine which of the following are linear transformation:

   1- $T : R^3 \to R^2$, $T(x_1, x_2, x_3) = (x_1x_2, x_3)$

   2- $S : R^3 \to R$, $S(x_1, x_2, x_3) = x_1 + x_2$

   3- $F : R^2 \to R^4$, $F(x_1, x_2) = (x_1^2, x_1 + x_2, x_2^2, x_2 + x_1)$

4. Let $Q, S, T : \mathbf{R^2} \to \mathbf{R^2}$ be linear transformations defined by;

   $Q(x_1, x_2) = (x_1 - x_2, x_1)$, $S(x_1, x_2) = (3x_2, 2x_2 - x_1)$, $T(x_1, x_2) = (x_1 + x_2, 3x_1 + 5x_1)$.

   Determine the following linear transformations:

   1) $2S + T$, 2) $T + Q - S$, 3) $2S - 4Q + 3T$.

5. Determine the composite $S \circ T$ of the following linear transformation. Then write down the matrices of T, S and $S \circ T$.

   a- $S(x_1, x_2) = (x_1 - x_2, x_2)$ and $T(y_1, y_2) = (2y_2, y_1)$

   b- $S(x_1, x_2) = (2x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$ and $T(y_1, y_2, y_3) = (y_1 - y_2, y_2 - y_3)$

6. Let $T, Q, S : \mathbf{R^2} \to \mathbf{R^2}$ be linear transformations defined by

   $$T(x_1, x_2) = (2x_2, x_1),\ Q(x_1, x_2) = (x_1 - x_2, x_2),\ S(x_1, x_2) = (3x_2, 2x_2 - x_1)$$

   Write down the matrix of T, S and Q.

7. Let $B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -9 & -1 \end{pmatrix}$

   Write down the linear transformation for these matrices.

8. A) Show that any linear transformation $T : \mathbf{R} \to \mathbf{R}$ has the form $T(x) = cx$ for some constant real number $c \in \mathbf{R}$.

   B) Show that any linear transformation $T : \mathbf{R} \to \mathbf{R}^n$ has the form $T(x) = \mathbf{v}x$ for some $\mathbf{v} \in \mathbf{R}^n$.

9. Let $||.||$ be a function on $\mathbf{R^3}$ defined by $||(x_1, x_2, x_3)|| = x_1 + x_2 + x_3$.

   Is $||.||$ a norm on $\mathbf{R^3}$, explain.

10. Let $(V, ||.||_1)$ and $(W, ||.||_2)$ be normed vector spaces.

    1- Show that $V \times W$ is a vector space.

    2- Define a function $||.||$ on $V \times W$ by

    $$||(v, w)|| = ||v||_1 + ||w||_2,\ (v, w) \in V \times W$$

    Show that this dfines a norm on $V \times W$.

11. A) Let $(V, ||.||)$ be a normed vector space. Define the open ball at v_0 with radius r, also define an open set $U \subset V$.

    B) let $\{v_n\}$ be a sequence in V. Write down what it means to say $\{v_n\}$ converges to a vector $v \in V$.

12. Let V be a normed vector space, let $\{v_n\}$ and $\{w_n\}$ be two convergent sequences such that $v_n \to v_0$ and $w_n \to w_0$. Show that

    1- the sequence $\{v_n + w_n\}$ is a convergent sequence that converges to $v_0 + w_0$

    2- the sequence $\{rv_n\}$ is a convergent sequence that converges to rv_0, $r \in \mathbf{R}$ and $r \neq 0$.

13. Let $\{v_n\}$ be a sequence in a normed vector space V. Show that $v \to v_0$ in the vector space V if and only if the sequence $\{||v_n - v_0||\}$ converges to 0 in $\mathbf{R}$.

KING SAUD UNIVERSITY
MATH 482
Problem sheet 2

Dr. Fatima Azmi

---

1. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that both $lim_{x\to 0} lim_{y\to 0} f(x,y)$ and $lim_{y\to 0} lim_{x\to 0} f(x,y)$ exist, but f is not continuous at 0.

2. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 1, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Does $lim_{x\to 0} lim_{y\to 0} f(x,y)$ and $lim_{y\to 0} lim_{x\to 0} f(x,y)$ exist ?. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exsit ?.

3. Using the defintion of continuity, show the following funtions are continuous:

1- $h : \mathbf{R}^2 \to \mathbf{R}$, $h(x,y) = 5x - 6y$

2- $g : \mathbf{R}^2 \to \mathbf{R}$, $g(x,y) = x^2 + 2y^2$

3- $f : \mathbf{R}^3 \to \mathbf{R}$, $f(x,y,z) = 2x + 3y - 4z$

4- $k : \mathbf{R}^3 \to \mathbf{R}$, $k(x,y,z) = xy + yz$

5- $p : \mathbf{R}^3 \to \mathbf{R}$, $p(x,y,z) = xyz$

4. Study the continuity of the following functions on $\mathbf{R}^2$:

$$a) - \ f(x,y) = \begin{cases} \frac{3\,x^2y^2}{x^2+4\,y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$b) - \ f(x,y) = \begin{cases} y^2 \ \sin(\frac{x}{y}), & y \neq 0 \\ 0, & y = 0 \end{cases}$$

$$c) - \ f(x,y) = \begin{cases} \frac{xy^4}{x^4+y^6}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

5. Let $F : \mathbf{R}^3 \to \mathbf{R}^2$ be given by $F(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$. Show F is continous if and only if both f_1 and f_2 are continuous.

6. Let $f : \mathbf{R}^k \to \mathbf{R}$ be continuous at v_0, suppose that $f(v_0) > a > 0$. Show that there exist $\delta > 0$ such that if $||v - v_0|| < \delta$, then $f(v) > a$.

7. Let $T : \mathbf{R}^n \to \mathbf{R}^p$ be a linear transformation. Show that there exist a positive number M such that $||T(v)|| \leq M||v||$ for all v. Then use this to show that T is continuous.

8. Show that $f(x) = x^{1/2}$ is continuous for all $x > 0$.

9. Show that $h(x, y) = (x^2 + 2xy + 3y^2 + 1)^{1/2}$ is continous for all x, y

10. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function. Prove that $g : \mathbf{R}^k \to \mathbf{R}$ defined by $g(v) = (f(v))^2$ is continuous, when f is continous, can g be continous when f is not continuous.

11. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a continous function. Let $\Gamma = \{(x, f(x)|x \in \mathbf{R}^n\}$ be the graph of f in $\mathbf{R}^n \times \mathbf{R}^m$. Prove that Γ is closed.

12. Let $f, g : \mathbf{R}^n \to \mathbf{R}$ be continuous functions at $v_0 \in \mathbf{R}^n$. Show that the product $f\,g$ is continuous at v_0.

13. Let $f : \mathbf{R}^m \to \mathbf{R}^p$ be a function. Show that f is continuous at v_0 if and only if whenever $\{v_n\}$ is a convergent sequence in $\mathbf{R}^m$ such that $v_n \to v_0$, then $f(v_n) \to f(v_0)$.

KING SAUD UNIVERSITY
MATH 482
Problem sheet 3

Dr. Fatima Azmi

---

1. Let $f : R^n \to R^m$ be a mapping, show the following are all equivalent:

   1- f is continuous.

   2- $f^{-1}(U)$ is open in R^n for every open set U in R^m.

   3- $f^{-1}(F)$ is closed in R^n for every closed set F in R^m.

2. A subset K of R^n is compact if and only if it is closed and bounded (Heine-Borel Theorem).

   Let $f : R^n \to R^m$ be a continuous map and let K be a compact susbet of R^n. Show that $f(K)$ is compact subset of R^m.

3. Let $T : R^n \to R^p$ be a linear transformation. We say T is a bounded linear transformation if there exist $M > 0$ such that $||T(v)|| \leq M||v||$ for all $v \in R^n$.

   Show that every linear transformation $T : R^n \to R^m$ is bounded.

4. Prove that every linear transformation $T : R^n \to R^m$ is uniformly continuous.

5. Let K be a compact subset of R^n. Show that every continuous function $f : K \to R^m$ is uniformly continuous.

6. Show that $f : R \to R$ defined by $f(x) = x^2$ is not uniformaly continuous, but if $f : [a, b] \to R$, then f is uniformly continuous.

7. Let $f : [a, b] \times [c, d] \to R$, defined by $f(x, y) = xy$. Is f uniformly continuous?

8. Let $f : R^2 \to R$ be defined by $f(x_1, x_2) = x_1^2 x_2$. Compute the directional derivative of f at v in the direction of v_0, where

   1- $v = (1, 0), v_0 = (0, 1)$

   2- $v = (-1, 2), v_0 = (1/2, \sqrt{3}/2)$

   3- $v = (1, 1), v_0 = (\sqrt{2}/2, \sqrt{2}/2)$

9. Let $f : R^3 \to R$ be defined by $f(x_1, x_2, x_3) = x_1x_2x_3$. Compute the directional derivative of f at v in the direction of v_0, where

   1- $v = (1, 0, 0), v_0 = (\sqrt{2}/2, 0, \sqrt{2}/2)$

   2- $v = (1, 1, -1), v_0 = (\sqrt{3}/2, -1/2, 0)$

   3- $v = (3, 2, 5), v_0 = (0, \sqrt{2}/2, \sqrt{2}/2)$

10. Let $g : R^2 \to R$ be defined by $g(x_1, x_2) = cos(x_1x_2)$. Compute the directional derivative of g at v in the direction of v_0, where

    1- $v = (1, 0), v_0 = (0, 1)$

    2- $v = (-1, 2), v_0 = (1/2, \sqrt{3}/2)$

    3- $v = (3, 2), v_0 = (\sqrt{2}/2, \sqrt{2}/2)$

11. Let $g : R^3 \to R$ be defined by $g(x_1, x_2, x_3) = sin(x_1x_3 - 2x_2^2)$. Compute the directional derivative of g at v in the direction of v_0, where

    1- $v = (1, 0, 0), v_0 = (\sqrt{2}/2, 0, \sqrt{2}/2)$

    2- $v = (1, 1, -1), v_0 = (\sqrt{3}/2, -1/2, 0)$

KING SAUD UNIVERSITY
MATH 482
Problem sheet 4

Dr. Fatima Azmi

---

1. Answer the following questions for the functions in part (a) and (b):

   1- For which vectors $u \neq 0$, does $f'(0; u)$ = directional derivative of f at 0 in the direction of u exist, evalute it when it exist.

   2- Does $\partial f/\partial x|_0$, $\partial f/\partial y|_0$ exist.

   3- Is f differentiable at $0 = (0,0)$?.

   4- Is f continuous at $0 = (0,0)$?.

   a) $f : R^2 \to R,\quad f(0) = 0,$

   $$f(x,y) = \frac{xy}{x^2+y^2}, \text{ if } (x,y) \neq (0,0)$$

   b) $f : R^2 \to R,\quad f(x,y) = |x| + |y|$

2. Let $f : R \to R,\quad f(0) = 0$ and $f(t) = t^2\, sin(1/t)$, for $t \neq 0$

   1- Show that f is differentiable at 0, compute $\partial f/\partial t|_0$

   2- Compute $\partial f/\partial t$, if $t \neq 0$

   3- Show that $\partial f/\partial t$ is not continuous at 0.

   4- Show that f is differentiable on R but f is not of class C^1 on R.

3. Let $f : R^2 \to R$, be defined by

   $$f(x,y) = \begin{cases} \frac{(xy)^2}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

   Show that f is differentiable at $(0,0)$.

4. Let $f : R^2 \to R$, be defined by $f(0,0) = 0$ and

   $$f(x,y) = \frac{x^2 y}{x^4+y^2}, \ (x,y) \neq (0,0)$$

   1- Show f has partial derivative every where

   2- Show that f is not continuous at $(0,0)$, is f differentiable at $(0,0)$.

5. Compute the Jacobian of the following:
   1- $f : R^3 \to R^2$, $f(x_1, x_2, x_3) = (x_1 e^{x_2}, x_2^2 + x_3 sin x_1)$
   2- $f : R^3 \to R^3$, $f(x_1, x_2, x_3) = (x_1 x_2, x_1^2 + x_2 x_3, x_2 x_3^2)$
   3- $f : R^2 \to R^2$, $f(x_1, x_2) = (x_1^2 + x_2, 2x_1 x_2 - x_2^2)$

6. Let $f(x, y) = x^2 + xy - y^2$, $F(x, y) = (xy^2, xy)$ and
   $g(x, y, z) = xyz^2$, $G(x, y) = (x + y, x - y, xy)$
   $h(x, y) = x siny$, $H(x, y, z) = (xyz, xe^{xy})$
   Compute the derivative of $(f \circ F)$, $(g \circ G)$, $(h \circ H)$, $(G \circ H)$ and $(F \circ F)$ using the chain rule.

7. Let $f, g : U \subset R^n \to R^m$ be both differentiable maps at at $v_0 \subset U$ where U is an open subset of R^n. Show that $f + g$ is differentaible at v_0 and

$$D_{v_0}(f + g) = D_{v_0} f + D_{v_0} g$$

   .

8. Let $f, g : R^n \to R^m$ be continous functions such that $f + g$ is differentaible. Are f and g differentiable ?, justify.

9. Let $f, g : R^n \to R^m$ be differentaible at v_0. Show that

$$\frac{\partial}{\partial x_j}(f_i + g_i) = \frac{\partial f_i}{\partial x_j} + \frac{\partial g_i}{\partial x_j}$$

   and

$$\frac{\partial (r\, f_i)}{\partial x_j} = r\, \frac{\partial f_i}{\partial x_j}, \text{ where r is some constant}$$

10. Let $f : R \to R$ be differentiable and define $g : R^2 \to R$ by $g(x, y) = f(x + y)$. Prove that $\partial g/\partial x - \partial g/\partial y = 0$

11. Let $f : R \to R$ be a function satisfying the condition,

$$|f(x) - f(y)| < M|x - y|^n$$

    for some $n > 1$. Prove that f is constant.

12. Let $f : R^n \to R^m$ be a mapping, suppose there exist M such that $||f(x)|| \leq M||x||^2$ for all x. Show f is differentiable at $x_0 = 0$ and $D_{x_0} f = 0$.

13. Let $L : R^n \to R^m$ be a linear transformation, let $g : R^n \to R^n$ be a mapping such that $||g(x)|| \leq M||x||^2$ for all x. Let $f(x) = L(x) + g(x)$. Prove that $D_0 f = L$.

KING SAUD UNIVERSITY
MATH 482
Problem sheet 5

Dr. Fatima Azmi

---

1. Compute the gradients and the direction where the directional derivative is largest of the following functions:

   1- $f(x_1, x_2) = x_1^2 x_2$ at $v = (1, 2)$

   2- $f(x_1, x_2, x_3) = cos(x_1 x_2 x_3)$ at $v = (1/2, -1/2, \pi)$

2. Let v_0 be a fixed vector, define $f : R^n \to R$ by $f(v) =< v, v_0 >$. Prove that f is continuous.

3. Let v and w be non zero vectors in R^n. Prove that

$$||u + v||^2 = ||u||^2 + ||v||^2 \iff \text{ u and v are orthogonal}$$

4. Show that if w_1 and w_2 are vectors in R^n such that for every $v \in R^n$,

   $< v, w_1 >=< v, w_2 >$ then $w_1 = w_2$.

5. Show that if $T : R^n \to R$ is a linear transformation then there exists a unique vector $w \in R^n$ such that $T(v) =< v, w >$.

6. Let $w \in R^n$ be a fixed vector, define $f : R^n \to R$ by $f(v) =< v, w >$. Prove that f is differentiable.

7. Find the equation of the hyperplane in R^m, with normal n containig the point b, where;

   1- $n = (0, 1, 1)$, $b = (2, -1, 3)$

   2- $n = (-4, 0, 2, 2)$, $b = (10, 7, -7, 14)$

   3- $n = (2, -3, 2, 1, 6)$, $b = (5, 1, 0, 1, -1)$

8. Let $f, g : R^n \to R$ be a differentiable functions and a, b real numbers. Prove that

$$grad(af + bg) = a\, grad(f) + b\, grad(g)$$

9. Let $g : R^2 \to R^2$ and $f : R^2 \to R$ be differentaible functions. Use the chain rule to derive a formula for $grad(f \circ g)$.

10. Find the equation of the tangent plane to the graph of the function f at the point $(v_0, f(v_0))$, where

1- $f : R^2 \to R, f(x_1, x_2) = 2x_1 + 5x_2 - 7,\ v_0 = (1, -1)$

2- $f : R^2 \to R, f(x_1, x_2) = x_1^2 x_2^5,\ v_0 = (2, -1)$

3- $f : R^3 \to R, f(x_1, x_2, x_3) = x_2 sin(x_1 x_3),\ v_0 = (1, 0, -2)$

4- $f : R \to R^2, f(x_1) = (2x_1 - 3, x_1 + 5),\ v_0 = -7$

5- $f : R \to R^3, f(x_1) = (2cosx_1, 2sinx_1, x_1),\ v_0 = \pi/4$

6- $f : R^2 \to R^4, f(x_1, x_2) = (x_1^2, -x_1,\ x_1^2 + x_2,\ x_1 x_2^2 + 3),\ v_0 = (1, -3)$

11. In the following functions, compute the following $\partial^2 f/\partial x^2,\ \partial^2 f/\partial x \partial y,\ \ \partial^3 f/\partial x \partial y^2$ and $H(f)(0, 0)$:

1- $f(x, y) = x^2 y - y^3$

2- $f(x, y) = xsiny$

3- $f(x, y) = arctan(x + y)$

4- $f(x, y) = (xy + y^2)^{1/3}$

12. Compute $H(f)(v_0)$, of the following functions:

1- $f(x_1, x_2, x_3) = x_1 x_2 - x_3^3, v_0 = (0, 0, 0)$ and at $v_0 = (1, 3, -1)$

2- $f(x_1, x_2, x_3, x_4) = x_1 x_3 - x_2 x_4,\ v_0 = (0, 0, 0, 0)$ and at $v_0 = (2, -1, 1, 3)$

13. Let $f, g : R \to R$ be differentaible functions and define $h : R^2 \to R$ by $h(x, t) = f(x - at) + g(x + at)$, where a is constant, show that h satisfies

$$a^2 \partial^2 h/\partial x^2 = \partial^2 h/\partial t^2$$

14. Let $f : R^2 \to R$ be defined by $f(0, 0) = 0$ and

$$f(x, y) = x\, y \frac{x^2 - y^2}{x^2 + y^2},\ (x, y) \neq (0, 0)$$

Show that $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous, that $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ exist at every point, and that

$$\partial^2 f/\partial x \partial y|_{(0,0)} \neq \partial^2 f/\partial y \partial x|_{(0,0)}$$

KING SAUD UNIVERSITY
MATH 482
Problem sheet 6

Dr. Fatima Azmi

---

1. Determine the mth degree Taylor polynomials of the following functions $f(x)$ about a and the remainder:

   1- $f(x) = ln(1+x)$, $a = 0$ and m is arbitrary.

   2- $f(x) = cos(x)$, $a = 0$ and m is arbitrary. Also for $a = \pi/4$ and m is arbitrary.

   3- $f(x) = arctan(x)$, $a = 0$ and m is arbitrary.

   4- $f(x) = e^x$, $a = 2$ and $m = 5$.

   5- $f(x) = (1+x)^r$, $r > 0$, $a = 0$ and $m = 3$.

2. Determine the following with error less than 10^{-3}.

   1- $e^{0.3}$, 2- $cos(\pi/4)$, 3- $ln(1.2)$ 4- $\sqrt{2}$.

3. Write down the Taylor polynomial of degree m about $a \in R^2$, where

   1- $f(x_1, x_2) = 3 + x_1x_2 + x_2^3$, $a = (0, 0)$ and $m = 3$ also at $m = 57$.

   2- $f(x_1, x_2) = cos(x_1x_2)$, $a = (0, 0)$ and $m = 3$, then do the same problem with $a = (2, \pi/4)$ and $m = 2$.

   3- $f(x_1, x_2) = x_1\, tan(x_1x_2)$, $a = (2, \pi/8)$ and $m = 2$

   4- $f(x_1, x_2) = e^{x_1\, sinx_2}$, $a = (0, \pi/3)$ and $m = 2$.

4. Write down the Taylor polynomial of degree m about $a \in R^3$, where

   1- $f(x_1, x_2, x_3) = x_1x_2x_3$, $a = (0, 0, 0)$ and $m = 3$

   2- $f(x_1, x_2, x_3) = \sin(x_1x_2x_3)$, $a = (2, 0, 1)$ and $m = 3$

   3- $f(x_1, x_2, x_3) = x_1 \cos(x_2x_3 + x_1^2)$, $a = (0, 2, \pi/8)$ and $m = 2$

KING SAUD UNIVERSITY
MATH 482
Problem sheet 7

Dr. Fatima Azmi

---

1. Find all the critical points of the following functions:

   1- $f(x) = 2x^3 - 9x^2 + 12x$

   2- $f(x, y) = 2x + 3y - 6$

   3- $f(x, y) = 3 - x^2 - y^2$

2. For each of the following functionss, $(0, 0)$ is a critical point. Determine whether it is a local max., a local min, or neither:

   1- $f(x, y) = x^4 + y^4$

   2- $f(x, y) = x^4 - y^4$

   3- $f(x, y) = -x^4 - y^4$

3. Find the quadratic forms associated to these matrices and determine if they are positive definite, negative definite or indefinite.

4. $A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 0 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -5 \end{pmatrix}$

   $C = \begin{pmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 3 & 1 \\ 0 & 3 & 5 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$

5. Find the matrix A corresponding to each of the quadratic forms Q_A given below, and determine if they are positive definite, negative definite or indefinite:

   1- $Q_A(x_1, x_2) = x_1^2 + 6x_1x_2 - 3x_2^2$

   2- $Q_A(x_1, x_2) = 4x_1x_2 + x_2^2$

   3- $Q_A(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$

   4 - $Q_A(x_1, x_2) = 4x_1x_2 - 7x_1^2 - 3x_2^2$

   5 - $Q_A(x_1, x_2, x_3) = x_1^2 - 2x_2^2 + 4x_3^2$

   6 - $Q_A(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 + x_2^2 + 4x_2x_3 + 8x_3^2$

6. Find and classify the critical points of the following:

   1- $f(x, y) = x^2 - 4x + 2y^2 + 7$

   2- $f(x, y) = 3xy - x^2 - y^2$

   3- $f(x, y) = x^2 - 2xy + 1/3y^3 - 3y + 8$

   4- $f(x, y, z) = x^2 + y^2 + z^2 + xz + yz + 2x - 2y - 4z + 10$

7. Find three positive numbers x, y, z whose sum is 24 which maximize xy^2z^3.

8. Maximize $f(x, y, z) = 6z - x^2 - y^2$ subject to the condition $x + y + z = 3$

9. Find the point on the unit sphere $x^2 + y^2 + z^2 = 1$ which is furtherst from the point $(1, 2, -2)$

10. Find the min. value of the function $f(x, y) = x^2 + 8xy - 5y^2$ subject to the constraint $x^2 + y^2 = 25$

11. Find the max. value of the function $f(x, y, z) = x^2 + xy + y^2 + xz + z^2$ subject to the constraint $x^2 + y^2 + z^2 = 4$

12. Find the min. of the function $x^p/p + y^q/q$ subject to the costarint $xy = 1$ where p and q are positive numbers.

13. Find the extreme values of the following functions f subject to the given constraint:

    1- $f(x, y) = 3x - y$, with constarint $g(x, y) = x^2 + y^2 - 4 = 0$

    2- $f(x, y) = x + y$, with constarint $g(x, y) = 9x^2 + 4y^2 - 36 = 0$

    3- $f(x, y, z) = 2x - y + z$, with constarint $g(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$

    4- $f(x, y, z) = x + y + z$, with constarint $g(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$

    5- $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 4 = 0$

14. Find the extreme values of $f(x, y, z) = 2x + y + 2z$ on the curve of intersection of $x^2 + y^2 - 4 = 0$ and $x + z = 2$

15. Find the min. value of $f(x, y, z) = xy + yz$ subject to the constarint $x^2 + y^2 = 2$ and $x^2 + z^2 = 2$

16. Find the extreme values of the given function on the given set:

    1- $f(x, y) = x^2 + y^2 - 5$ on the set $E = \{(x, y) \in R^2 \ni 4x^2 + 9y^2 \leq 36\}$

    2- $f(x, y) = x^6 + 2y^2$ on the set $D = \{(x, y) \in R^2 \ni x^2 + y^2 \leq 1\}$

    3- $f(x, y, z) = x^4 + y^4 + z^4 + 5$ on the set $K = \{(x, y, z) \in R^3 \ni x^2 + 4y^2 + 4z^2 \leq 4\}$

KING SAUD UNIVERSITY
MATH 482
Problem sheet 8

Dr. Fatima Azmi

---

1. Let $T : R^n \to R^n$ be a linear transformation which has an inverse, show that T^{-1} is also a linear transformations.

2. Let $T : R^n \to R^n$ be a linear transformation with matrix A that represent T. Show that T is invertible if and only if A is invertible matrix

3. Which of the following functions has an inverse, determine the inverse function when it exist:

   a) $f : (2, 6) \to (10, 42),\ f(x) = x^2 + 6$

   b) $f : (-3, -1) \to (7, 15),\ f(x) = x^2 + 6$

   c) $f : R \to R,\ f(x) = x^m$, where m is odd.

   d) $f : (0, \pi) \to (-1, 1),\ f(x) = \cos x$

4. Determine the inverse linear transformation and its matrix for each of the following linear transformation:

   a) T: $\mathrm{R}^2 \to R^2$, $T(x, y) = (x + a\, y, y)$, $a \in R$

   b) T: $\mathrm{R}^3 \to R^3$, $T(x, y, z) = (z, x, y)$

5. Let $f : R^2 \to R^2, \quad f(x, y) = (e^x cosy, e^x siny)$

   Show that f is locally invertible near every point but f is not invertible.

6. Which of the following equations implicitly define z as a function of x and y near the given point? when the function does, compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point.

   1- $x^2 + y^2 + z^2 = 49$; $(6, -3, -2)$

   2- $xy + yz + xz - xyz - 4 = 0$; $(2, 2, -3)$

   3- $xye^z + zcos(x^2 + y^2) = 0$; $(0, 0, 0)$

   4- $xy^6 - yz^3 + z - xy - 11 = 0$; $(1, 2, 3)$

7. Consider the system of equations:

$$xu + yv^2 = 0, \text{ and } xv^3 + y^2u^6 = 0$$

   Show that u and v can be defined as functions of x and y near the point $(1, -1, 1, -1)$. Then compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ at this point.

8. Which of the following equations define z and w as functions of x and y near the given point (x, y, z, w)? when the function does, compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at the point.

   1) $z^2 + w^2 = x^2 + y,$ and $z + w = x^2 - y;\ (2, 1, 1, 2)$

   2) $x - y^2 + z^2 + 2w^2 = 6,$ and $x^2 + y^2 = z^2 + w^2;\ (1, 2, -1, 2)$

   3) $x^2 + y^2 - z + w = 3,$ and $xy + zw + 1 = 0;\ (1, 0, -1, 1).$

KING SAUD UNIVERSITY
MATH 482
Problem sheet 9

Dr. Fatima Azmi

For $a = (a_1, a_2, .., a_n)$ and $(b = (b_1, b_2, ..., b_n)$, then $R(a, b)$ denote the rectangle $\{(x_1, x_2, ..., x_n) \ni a_i \leq x_i \leq b_i, i = 1, ..n\}$ in R^n.

Answer the following questions:

1. Using the definition of the integral compute the following integral:

   1) $f(x, y) = 2x + y$, 2)$f(x, y) = x^2y$

   over the rectangle $R(a, b)$, where $a = (0, 0)$ and $b = (1, 1)$

   3) $f(x, y, z) = xyz$ over the rectangle $R(a, b)$, where $a = (0, 0, 0)$ and $b = (1, 1, 1)$.

2. Let $f : R(a, b) \subset R^3 \to R$ be the constant function, $f(v) = c$ for all $v \in R(a, b)$. Prove using the definition of the integral that

   $$\int_{R(a,b)} f = c\mu(R(a, b)), \text{ where } \mu(R(a, b)) \text{ is the volume of the rectangle } R(a, b)$$

3. Let $f : R(a, b) \subset R^2 \to R$ be the function defined by $f(x, y) = x$ for all $(x, y) \in R(a, b)$. Prove using the definition of the integral that

   $\int_{R(a,b)} f = 1/2$, where $a = (0, 0)$ and $b = (1, 1)$.

4. Let $f : R(a, b) \subset R^2 \to R$ be the function defined by $f(x, y) = x$ for all $(x, y) \in R(a, b)$. Prove using the definition of the integral that

   $\int_{R(a,b)} f = 1/2(b_2 - a_2)(b_1^2 - a_1^2)$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

5. Let $f : R(a, b) \subset R^4 \to R$ be the function defined by $f(x_1, x_2, x_3, x_4) = x_1$ for all $(x_1, x_2, x_3, x_4) \in R(a, b)$. Prove using the definition of the integral that

   $$\int_{R(a,b)} f = 1/2(b_4 - a_4)(b_3 - a_3)(b_2 - a_2)(b_1^2 - a_1^2)$$

   where $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$.

KING SAUD UNIVERSITY
MATH 482
Problem sheet 10

Dr. Fatima Azmi

---

1. Find $\int_{R(a,b)} f$ where:

   1) $f(x,y) = y - 2x$, $a = (1,3)$, $b = (2,5)$

   2) $f(x,y) = y\sin(xy)$, $a = (0,0)$, $b = (\pi/4, 1)$

   3) $f(x,y,z) = yz^2(1+x^2)^{-1}$, $a = (0,-3,0)$, $b = (1,0,2)$

2. Compute the integrals

   1) $\int_S (1-x^2)(1+y^2)dA$, and the inetgral 2) $\int_S x\sin y dA$

   where S is the region bounded by the curves $y = 1 - x^2$ and $y = x^2$.

3. Compute the integral of f over S, where

   1) $f(x,y,z) = x + y^2 - xy$, and S is the region bounded by the xy- plane, the plane $z = 2$ and the cylinder $x^2 + y^2 = 1$.

   2) $f(x,y,z) = 1$ and S is the region bounded by the surface $z = 3x^2$, $z = 4 - x^2$, $y + z = 6$ and $y = 0$.

4. Find the volume of the region bounded by two cylinders $x^2 + y^2 = 9$ and $y^2 + z^2 = 9$

5. Let T be the transformation from xy plane to uv plane defined by $T(x,y) = (u,v)$ where $u = 3x$ and $v = 5y$.

   1) Find T^{-1}

   2) Find the curve in the xy plane that gets transformed via T where

   a) the circle $x^2 + y^2 = 1$

   b) the rectangle with vertices $(0,0), (0,1), (2,1), (2,0)$

6. Let T be the transformation from xy plane to uv plane defined by $T(x,y) = (u,v)$ where $u = -5x + 4y$ and $v = 2x - 3y$.

   1) Find T^{-1}

   2) Find the curve in the xy plane that gets transformed via T where

   a) the circle $x^2 + y^2 = 1$

   b) the square with vertices $(0,0), (1,-1), (2,0), (1,1)$

7. Evaluate the integral $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}(x^2+y^2)^{3/2}dydx$, using polar coordinates.

8. Compute the integral
$$\int\int_{\mathcal{R}}\frac{dA}{(1+x^2+y^2)^{3/2}}$$
where $\mathcal{R}$ is the region bounded by the curves $y=0$, $x=1$ and $y=x$.

9. **A-** The Cylinderical coordinates in R^3 are defined by $(x,y,z)=h(r,\theta,z)=(r\cos\theta,r\sin\theta,z)$. Show that $|det(Dh)|=|r|$.

   B- Use the Cylinderical coordinates to compute the integral of $f(x,y,z)=z(1+x^2+y^2)^{-1}$ over the region S, where

   1 - $S=\{(x,y,z)\ni 1\leq x^2+y^2\leq 3, 1\leq z\leq 5\}$

   2 - $S=\{(x,y,z)\ni 1\leq x^2+y^2\leq 3, x\geq 0, y\geq 0,\ 1\leq z\leq 5\}$.

10. Use the Cylinderical coordinates to compute the integral $\int\int\int_Q(x^2+y^2)dv$, where Q is the region bounded from above by the cylinder $x^2+y^2=1$ and by the two planes $z=0$ and $z=4$.

11. The Spherical coordinates in R^3 are defined by
$(x,y,z)=h(\rho,\theta,\phi)=(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)$. Show that $|det(D(h)|=\rho^2|\sin\phi|$

12. Use the Spherical coordinates to compute the following integrals:

    1- $\int\int\int_Q(x^2+y^2)dv$, where $Q=\{(x,y,z)\ni x^2+y^2+z^2\leq 1\}$

    2- $\int\int\int_Q x^2dv$, where Q is the region bounded between the two spheres $x^2+y^2+z^2=4$ and x$^2+y^2+z^2=9$.

13. Evaluate the integral after making the indicated change of variables

    1) $\int\int_R(x-y)^2\cos^2(x+y)dxdy$

    where boundary of R is : the square with vertices $(0,1),(1,2),(2,1),(1,0)$ and the change of variables is $u=x-y$, $v=x+y$

    2) $\int\int_R(x^2+2y^2)dxdy$

    where boundary of R is : $xy=1, xy=2, y=|x|, y=2x$ and the change of variables is $x=u/v$, $y=v$

    3) $\int\int_R e^{-(x^2+y^2)}dxdy$

    where boundary of R is the region in the first quadrant defined by $R=\{(x,y)\ni 1\leq x^2+y^2\leq 2,\ x\geq 0,\ y\geq 0\}$

==