The Fourier Transform of 2-D Sequences

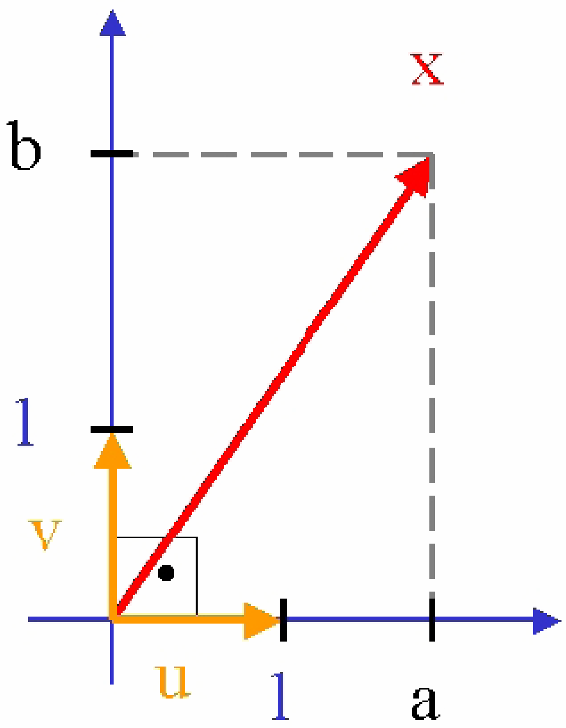
We will now review the Fourier Transform of 2-D sequences.

Motivation:

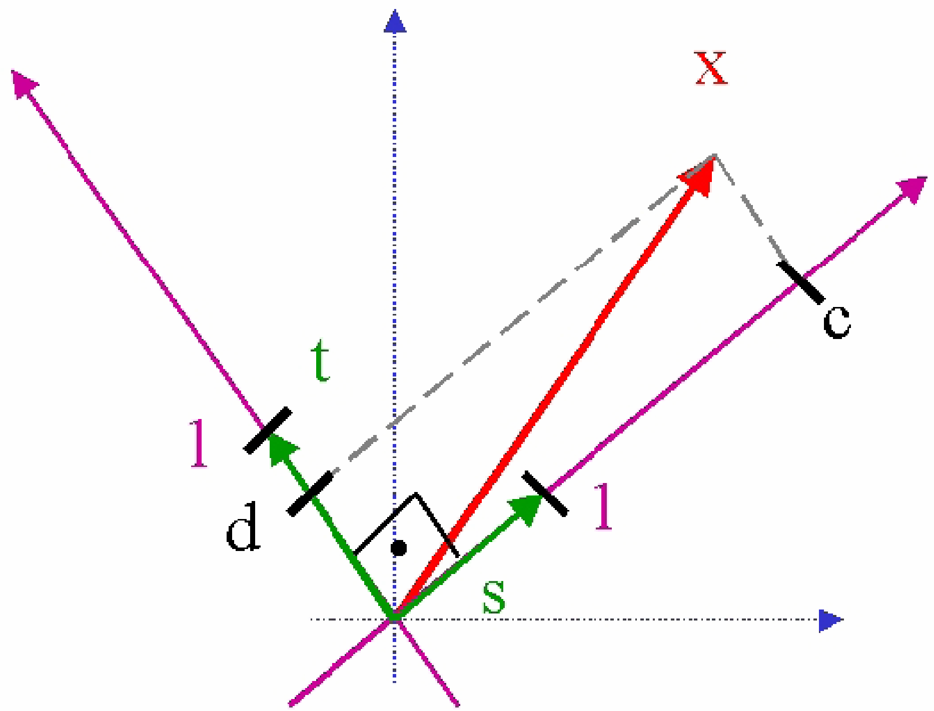
- The convolution operation takes on a very special form in 2-D Fourier transform "domain".
- The 2-D Fourier transform of images will reveal interesting properties that are shared by many images.
  - This will allow us to distinguish natural images from "non-images" (such as noise).
  - We will be able to say what "kind" of linear filter is good for a certain processing application.
- The effect of sampling operations are understood more clearly in 2-D Fourier transform "domain".

Intuition - Orthogonal Coordinate Systems

Original Coordinate System



Rotated Coordinate System



u , v , s , t : unit vectors

$$x = a u + b v = c s + d t$$

Definition

The 2-D Fourier Transform of a 2-D sequence $\mathbf{A}$, $\mathcal{F}(\mathbf{A})$ is defined as:

$$\begin{aligned}\mathcal{F}(\mathbf{A}) &= F_A(w_1, w_2) \\ &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A(m,n) e^{-j(mw_1+nw_2)} \qquad -\pi \leq w_1, w_2 < \pi\end{aligned}$$

$\mathbf{A}$ can be *recovered* back from its transform $F_A(w_1, w_2)$ via the inverse 2-D Fourier Transform $\mathcal{F}^{-1}(\mathbf{A})$:

$$\begin{aligned}A(m,n) &= \mathcal{F}^{-1}(\mathbf{A}) \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F_A(w_1, w_2) e^{+j(mw_1+nw_2)} dw_1 dw_2\end{aligned}$$

- w_1, w_2 vary in a continuum, i.e., the interval $[-\pi, \pi)$.
- $e^{j(mw_1+nw_2)} = cos(mw_1 + nw_2) + j sin(mw_1 + nw_2)$.
- $\mathbf{A} \stackrel{\mathcal{F}}{\leftrightarrow} F_A$

Real-Complex Parts and Symmetry

- In general $F_A(w_1, w_2)$ is complex valued.
- Since we will be mainly be considering real 2-d sequences we can note some symmetry properties by using the inverse Fourier transform relationship.

$$A(m,n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1, w_2) e^{+j(mw_1+nw_2)} dw_1 dw_2$$

If $\mathbf{A}$ is real then:

$$\begin{aligned}
F_A(w_1, w_2) &= F_A^*(-w_1, -w_2) \\
|F_A(w_1, w_2)| &= |F_A(-w_1, -w_2)| \\
\angle\, F_A(w_1, w_2) &= -\angle\, F_A(-w_1, -w_2) \\
\Re(F_A(w_1, w_2)) &= \Re(F_A(-w_1, -w_2)) \\
\Im(F_A(w_1, w_2)) &= -\Im(F_A(-w_1, -w_2))
\end{aligned}$$

Periodicity

$$F_A(w_1, w_2) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A(m, n) e^{-j(mw_1 + nw_2)} \qquad -\pi \leq w_1, w_2 < \pi$$

- $F_A(w_1, w_2)$ is periodic in w_1, w_2 with period 2π, i.e., for all integers k, l:

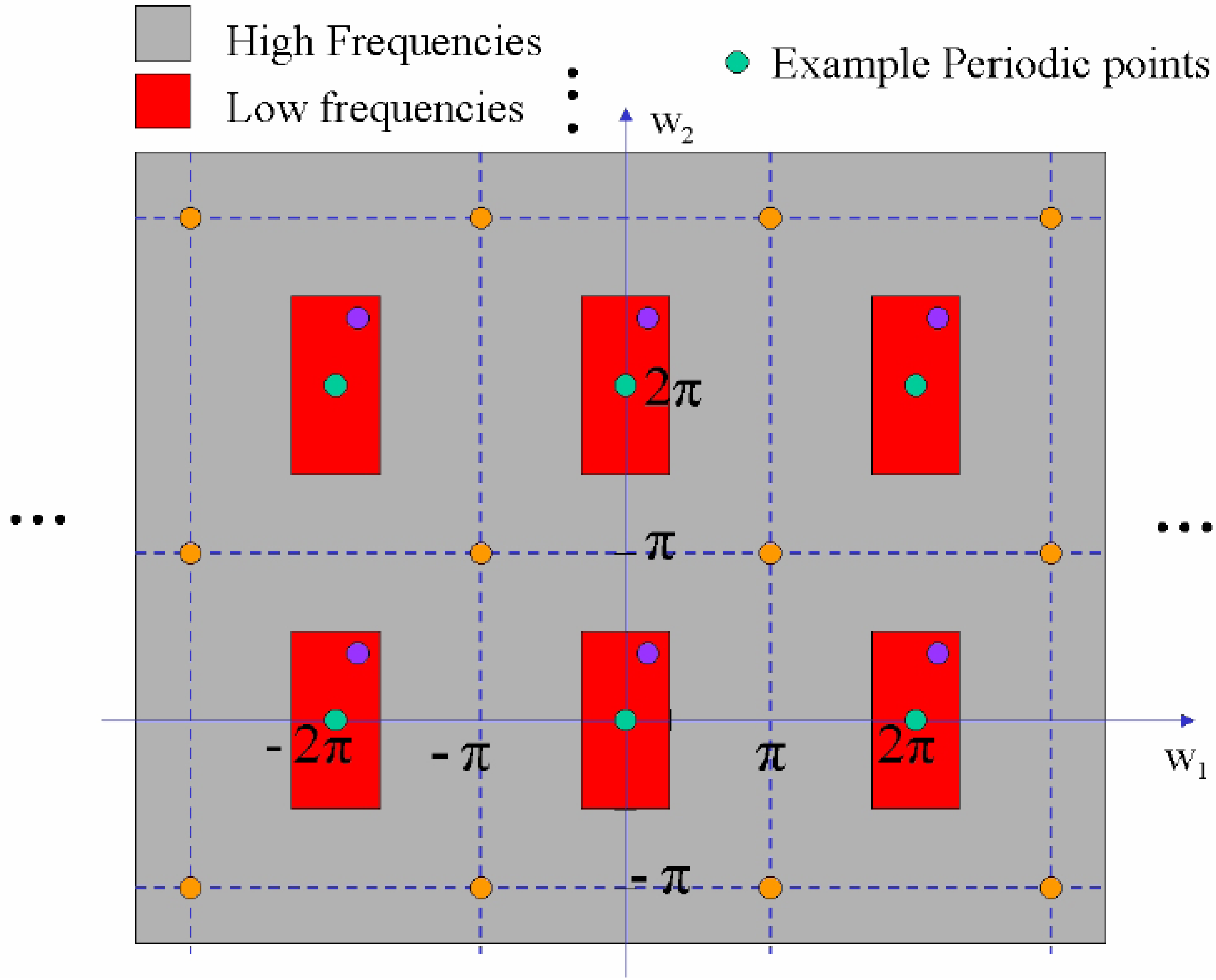
$$F_A(w_1 + k2\pi, w_2 + l2\pi) = F_A(w_1, w_2)$$

  To see this consider:

$$\begin{aligned} e^{-j(m(w_1+k2\pi)+n(w_2+l2\pi))} &= e^{-j(mw_1+nw_2)} e^{-jk2\pi} e^{-jl2\pi} \\ &= e^{-j(mw_1+nw_2)} \quad \forall \quad \text{integers} \quad k, l \end{aligned}$$

- $e^{j(mw_1+nw_2)} = cos(mw_1 + nw_2) + j sin(mw_1 + nw_2)$.
  w_1, w_2 the frequencies of the periodic trigonometric functions.

Example



High Frequencies
Low frequencies
Example Periodic points
w2
2π
π
-2π
- π
π
2π
- π
w1

Shifting and Modulation

- Shifting:

$$\begin{aligned}\mathcal{F}(A(m-m_0,n-n_0)) &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m-m_0,n-n_0)e^{-j(mw_1+nw_2)} \\ &= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} A(k,l)e^{-j((k+m_0)w_1+(l+n_0)w_2)} \\ &= e^{-j(m_0w_1+n_0w_2)}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} A(k,l)e^{-j(kw_1+lw_2)}\end{aligned}$$

$A(m-m_0,n-n_0) \stackrel{\mathcal{F}}{\leftrightarrow} e^{-j(m_0w_1+n_0w_2)}F_A(w_1,w_2)$.

- Similarly, modulation:

$$e^{j(mw_{01}+mw_{02})}A(m,n) \stackrel{\mathcal{F}}{\leftrightarrow} F_A(w_1-w_{01},w_2-w_{02})$$

Convolution

- Let $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$.

$$
\begin{aligned}
C(m,n) &= \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l) B(m-k, n-l) \\
F_C(w_1,w_2) &= \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l) \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} B(m-k, n-l) e^{-j(mw_1+nw_2)} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l) F_B(w_1,w_2) e^{-j(kw_1+lw_2)} \\
&= F_B(w_1,w_2) \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l) e^{-j(kw_1+lw_2)} \\
&= F_A(w_1,w_2) F_B(w_1,w_2)
\end{aligned}
$$

where we used the shifting property in the second step of the calculation. Thus we have the important result:

$$
\mathbf{A} \otimes \mathbf{B} \overset{\mathcal{F}}{\leftrightarrow} F_A(w_1,w_2) F_B(w_1,w_2)
$$

Delta Functions

- The Fourier transform of a Kronecker delta function:

$$F_\delta(w_1, w_2) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(m, n) e^{-j(mw_1+nw_2)} = 1$$

- The Fourier transform of $A(m,n) = 1$ can be found via the Dirac delta function:

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \delta(w_1, w_2) e^{j(mw_1+nw_2)} dw_1 dw_2 = \frac{1}{4\pi^2}$$

$$\Rightarrow A(m,n) = 1 \overset{\mathcal{F}}{\leftrightarrow} 4\pi^2 \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(w_1 - k2\pi, w_2 - l2\pi)$$

where $\delta(w_1, w_2)$ is the Dirac delta function and we used the fact that the Fourier transform has to be periodic with 2π.

- Note that $\delta(w_1, w_2) = 0$ for $w_1, w_2 \neq 0$ and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(w_1, w_2) dw_1 dw_2 = 1$$

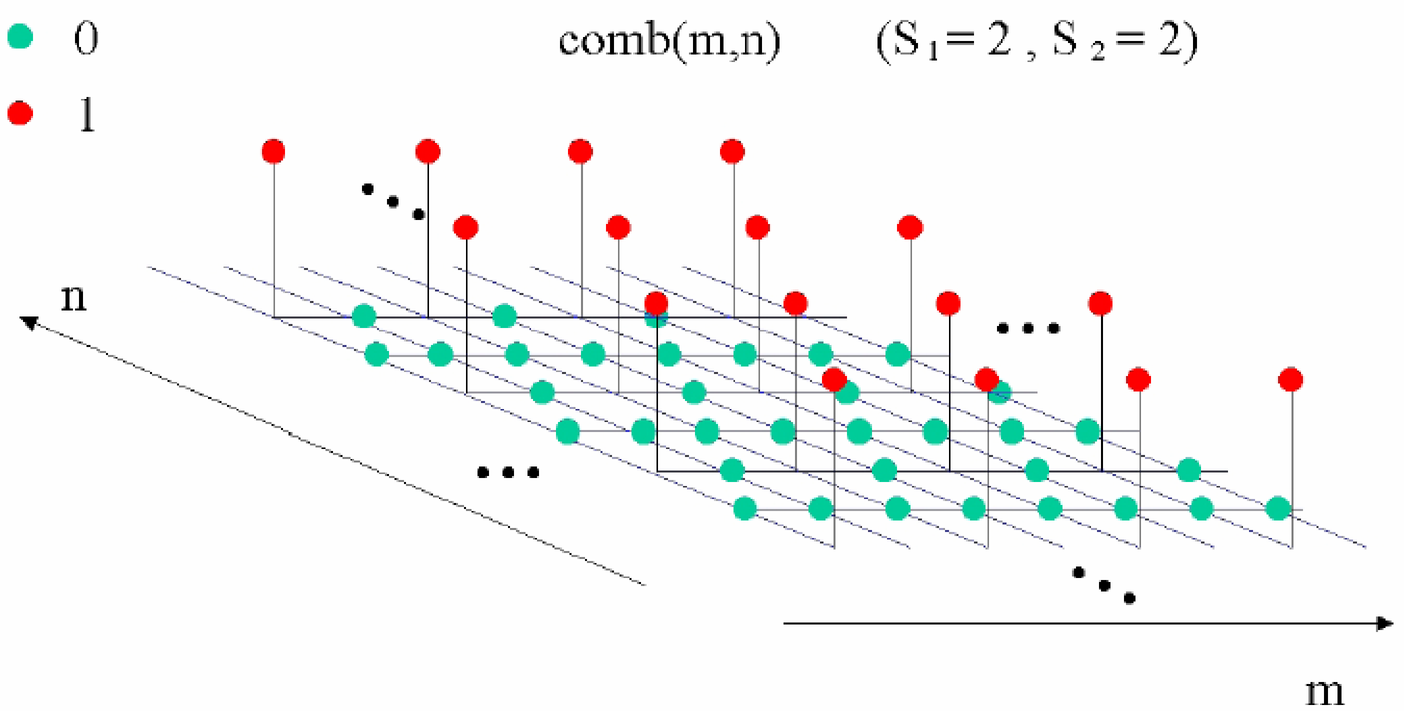
comb(m,n)

- Consider the Kronecker comb function $comb(m,n)$:

$$\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(m - kS_1, n - lS_2)$$

  where $S_1 > 0$, $S_2 > 0$ are integers.

- $comb(m,n)$ is very useful when discussing sampling.



comb(m,n) - contd.

The Fourier transform of a comb function can be computed as:

$$
\begin{aligned}
\mathcal{F}(comb(m,n)) &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} comb(m,n) e^{-j(mw_1+nw_2)} \\
&= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} [\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(m-kS_1, n-lS_2)] e^{-j(mw_1+nw_2)} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} [\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(m-kS_1, n-lS_2) e^{-j(mw_1+nw_2)}] \\
&= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} 1 \; e^{-j(kS_1w_1+lS_2w_2)} \\
&= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} 1 \; e^{-j(mS_1w_1+nS_2w_2)}
\end{aligned}
$$

comb(m,n) - contd.

Note that Equation 20 is simply the Fourier transform of $A(m,n) = 1$ and hence:

$$\begin{aligned}\mathcal{F}(comb(m,n)) &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} 1\ e^{-j(mS_1w_1+nS_2w_2)} \\ &= 4\pi^2 \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} \delta(S_1w_1 - k2\pi, S_2w_2 - l2\pi) \\ &= \frac{4\pi^2}{S_1S_2}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} \delta(w_1 - \frac{k2\pi}{S_1}, w_2 - \frac{l2\pi}{S_2})\end{aligned}$$

where the last line follows since for any "regular" function $G(w_1, w_2)$:

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(S_1w_1 - k2\pi, S_2w_2 - l2\pi)G(w_1,w_2)dw_1dw_2 &= \frac{1}{S_1S_2}G(k2\pi/S_1, l2\pi/S_2) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{S_1S_2}\delta(w_1 - \frac{k2\pi}{S_1}, w_2 - \frac{l2\pi}{S_2})G(w_1,w_2)dw_1dw_2\end{aligned}$$

and Dirac delta functions are defined by integrals.

Fourier Transform Types

Let $0 < a_1 < b_1 < \pi$ and $0 < a_2 < b_2 < \pi$.

- We will say that a Fourier transform $F_A(w_1, w_2)$ is low pass if $|F_A(w_1, w_2)| \sim 0$ when $a_1 < |w_1| < \pi$ and $a_2 < |w_2| < \pi$.
- We will say that a Fourier transform $F_A(w_1, w_2)$ is high pass if $|F_A(w_1, w_2)| \sim 0$ when $0 < |w_1| < a_1$ and $0 < |w_2| < a_2$.
- Finally, we will say that a Fourier transform $F_A(w_1, w_2)$ is band pass if $|F_A(w_1, w_2)| \sim 0$ when $0 < |w_1| < a_1, \ b_1 < |w_1| < \pi$ and $0 < |w_2| < a_2, \ b_2 < |w_2| < \pi$.

Sampling and Aliasing

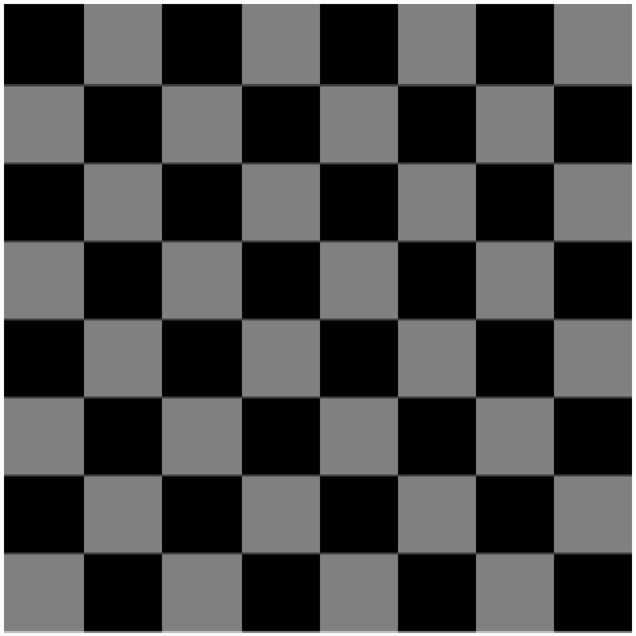
- Given a 2-D sequence $\mathbf{A}$ we would like to obtain a sequence $\mathbf{C}$ by sub-sampling $\mathbf{A}$:

$$C(m,n) = A(S_1 m, S_2 n)$$

where $S_1, S_2 > 0$ are integers.

- We would like $\mathbf{C}$ to have close resemblance to $\mathbf{A}$.
- For example given a 512×512 image we would like to obtain a 256×256 image by picking every other pixel in the original image.
- Things may go very wrong in sampling with unexpected effects.

Example



Original image (8x8)

Two possible 4x4 sub-sampled images

Fourier Transform of Sampled Sequence

$$\begin{aligned} B(m,n) &= A(m,n)comb(m,n) \\ C(m,n) &= B(S_1 m, S_2 n) \end{aligned}$$

- First obtain the Fourier transform of B using the multiplication property:

$$F_B(w_1,w_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1',w_2')F_{comb}(w_1-w_1',w_2-w_2')dw_1'dw_2'$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1',w_2')[\frac{4\pi^2}{S_1S_2}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\delta(w_1-w_1'-\frac{k2\pi}{S_1},w_2-w_2'-\frac{l2\pi}{S_2})]dw_1'dw_2'$$

$$= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{1}{S_1S_2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1',w_2')\delta(w_1-w_1'-\frac{k2\pi}{S_1},w_2-w_2'-\frac{l2\pi}{S_2})dw_1'dw_2'$$

For $w_1, w_2 \in [-\pi,\pi)$, let $K(w_1) = \{k | w_1 - \frac{k2\pi}{S_1} \in [-\pi,\pi)\}$ and $L(w_2) = \{l | w_2 - \frac{l2\pi}{S_2} \in [-\pi,\pi)\}$. Then:

$$F_B(w_1,w_2) = \frac{1}{S_1S_2}\sum_{k\in K(w_1)}\sum_{l\in L(w_2)} F_A(w_1-\frac{k2\pi}{S_1},w_2-\frac{l2\pi}{S_2})$$

Example

Suppose $S_1 = S_2 = 2$, i.e., we are sub-sampling by 2. Then for $w_1, w_2 \in (-\pi, \pi)$,

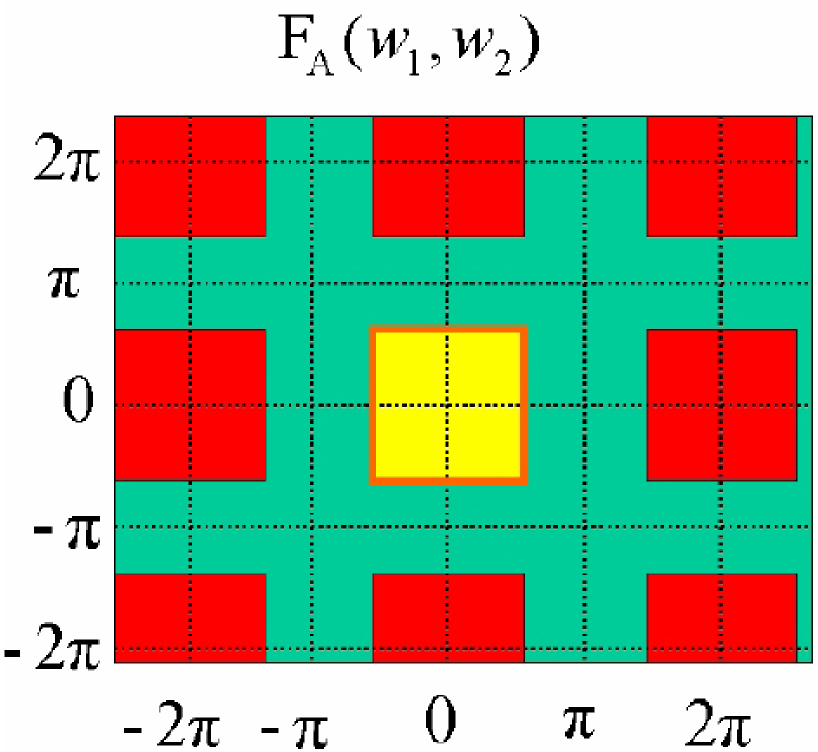
$$\begin{aligned} K(w_1) &= \{k | w_1 - k\pi \in (-\pi, \pi)\} = \{-1, 0, 1\} \\ L(w_2) &= \{l | w_2 - l\pi \in (-\pi, \pi)\} = \{-1, 0, 1\} \end{aligned}$$

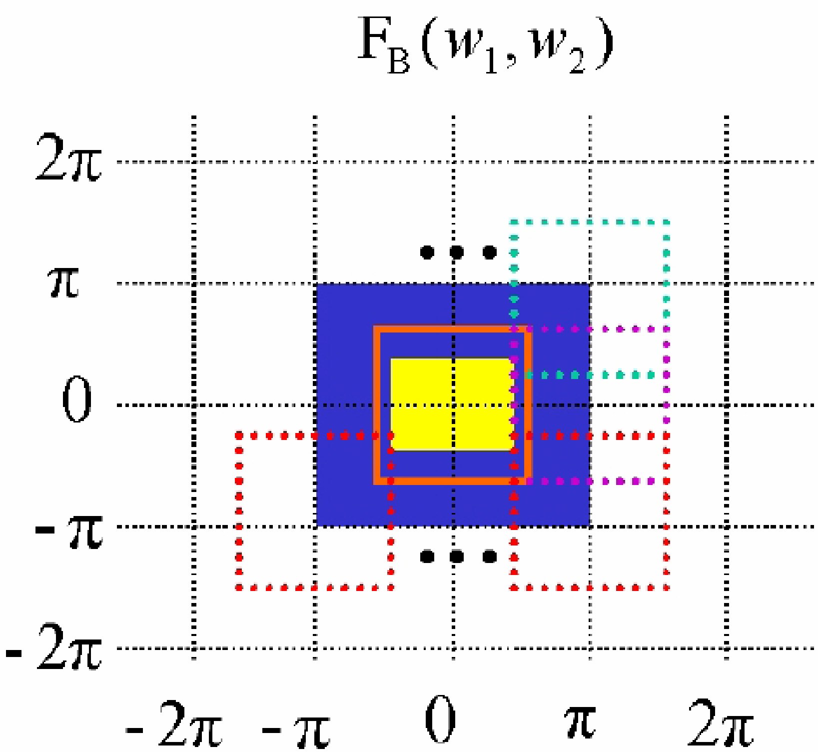
and we have:

$$F_B(w_1, w_2) = \sum_{k=-1}^{1} \sum_{l=-1}^{1} F_A(w_1 - k\pi, w_2 - l\pi)$$

- If during this process there is overlapping, i.e., say $(F_B(w_1, w_2) - F_A(w_1, w_2))F_A(w_1, w_2) \neq 0$ then we will say that there is aliasing in the sub-sampling operation.

Example - contd.





Aliased Frequencies

Aliased frequencies shown inside $[-\pi, \pi) \times [-\pi, \pi)$ only

F-T of Sampled Sequence - contd.

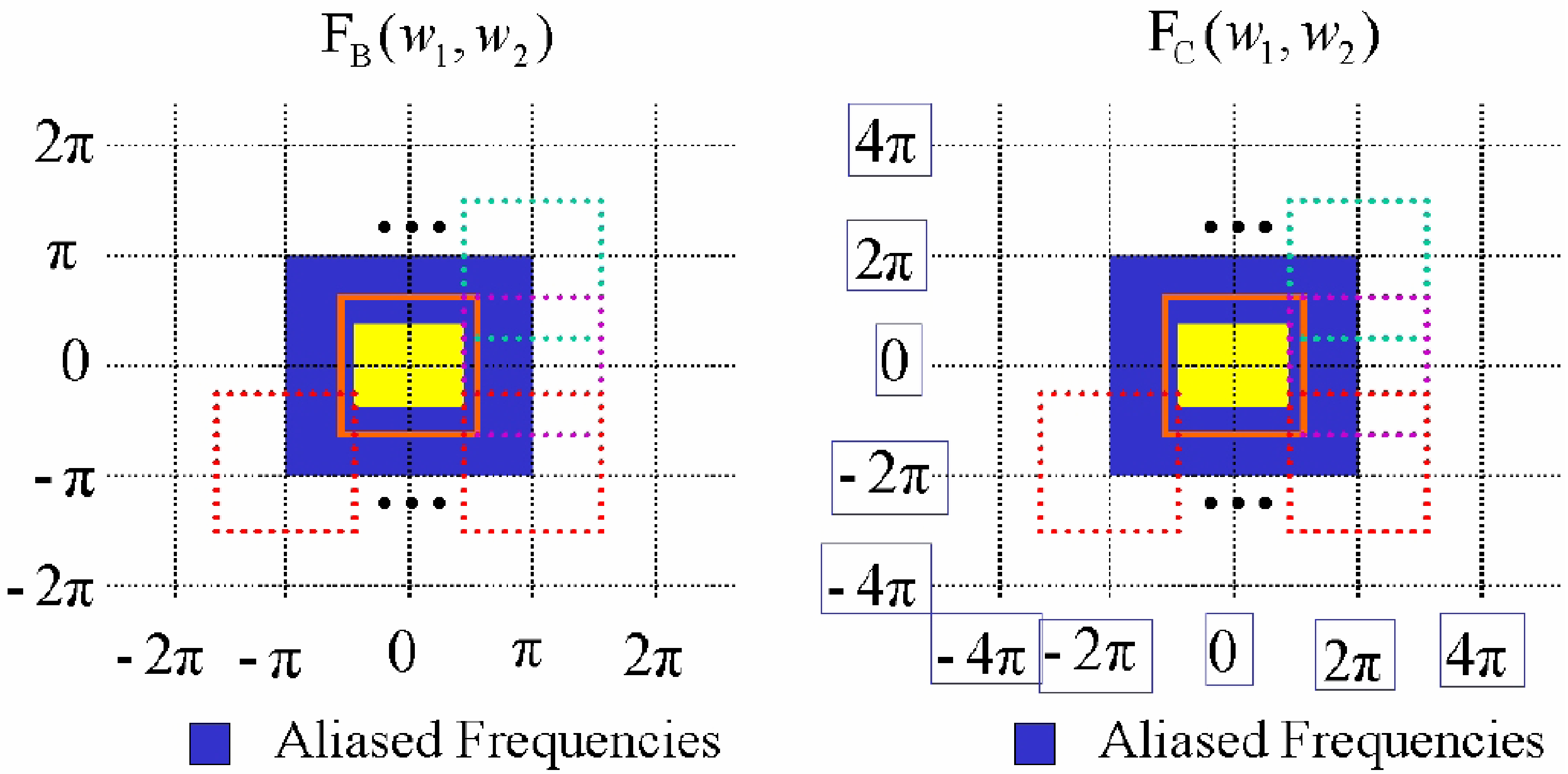
Going back to the transform we were calculating:

- We can now obtain $F_C(w_1, w_2)$

$$\begin{aligned}
F_C(w_1,w_2) &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} B(S_1m, S_2n)e^{-j(mw_1+nw_2)} \\
&= \sum_{m=\ldots,-S_1,0,S_1,\ldots}\;\sum_{n=\ldots,-S_2,0,S_2,\ldots} B(m,n)e^{-j(mw_1/S_1+nw_2/S_2)} \\
&= F_B(w_1/S_1, w_2/S_2) \\
&= \frac{1}{S_1S_2}\sum_{k\in K(w_1)}\sum_{l\in L(w_2)} F_A(\frac{w_1}{S_1}-\frac{k2\pi}{S_1},\frac{w_2}{S_2}-\frac{l2\pi}{S_2})
\end{aligned}$$

Example - contd.

$$S_1 = S_2 = 2$$



Aliasing

- It is clear that unless we are careful, the sampled signal can be very different from the original.
- For no aliasing to occur after sampling $F_A(w_1, w_2)$ must be:

$$F_A(w_1, w_2) = 0, \ \frac{\pi}{S_1} < |w_1| < \pi, \ \frac{\pi}{S_2} < |w_2| < \pi$$

  so that there is no overlap.
- But what if there is?
  - Then we have to low-pass filter $\mathbf{A}$ to make sure things become conformant to the above.
  - Such a low-pass filter is called an antialiasing filter.

Homework - Lecture 4

1. Show the modulation property of the Fourier transform.

2. Show that if a two dimensional sequence is separable then so is its Fourier transform, i.e., if $A(m,n) = A_1(m)A_2(n)$ then $F_A(w_1, w_2) = F_{A_1}(w_1)F_{A_2}(w_2)$ where $F_{A_1}(w_1), F_{A_2}(w_2)$ are *one* dimensional Fourier transforms such as $F_a(w) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} a(n)e^{-jwn}$.

3. Obtain the Fourier transform of the 2-D sequence $A(m,n)$ given by:

	$n=0$	1	2
$m=0$	1	2	-1
1	2	4	-2
2	-1	-2	1

Simplify your answer as much as possible.

4. Calculate the Fourier transform of the limited extent sequence $A(m,n) = 1,\ 0 \leq m < 8, 0 \leq n < 8$ and $A(m,n) = 0$ otherwise.

5. Calculate the Fourier transform of the limited extent sequence $A(m,n) = (-1)^{m+n},\ 0 \leq m < 8, 0 \leq n < 8$ and $A(m,n) = 0$ otherwise.