Fourier Transform Properties

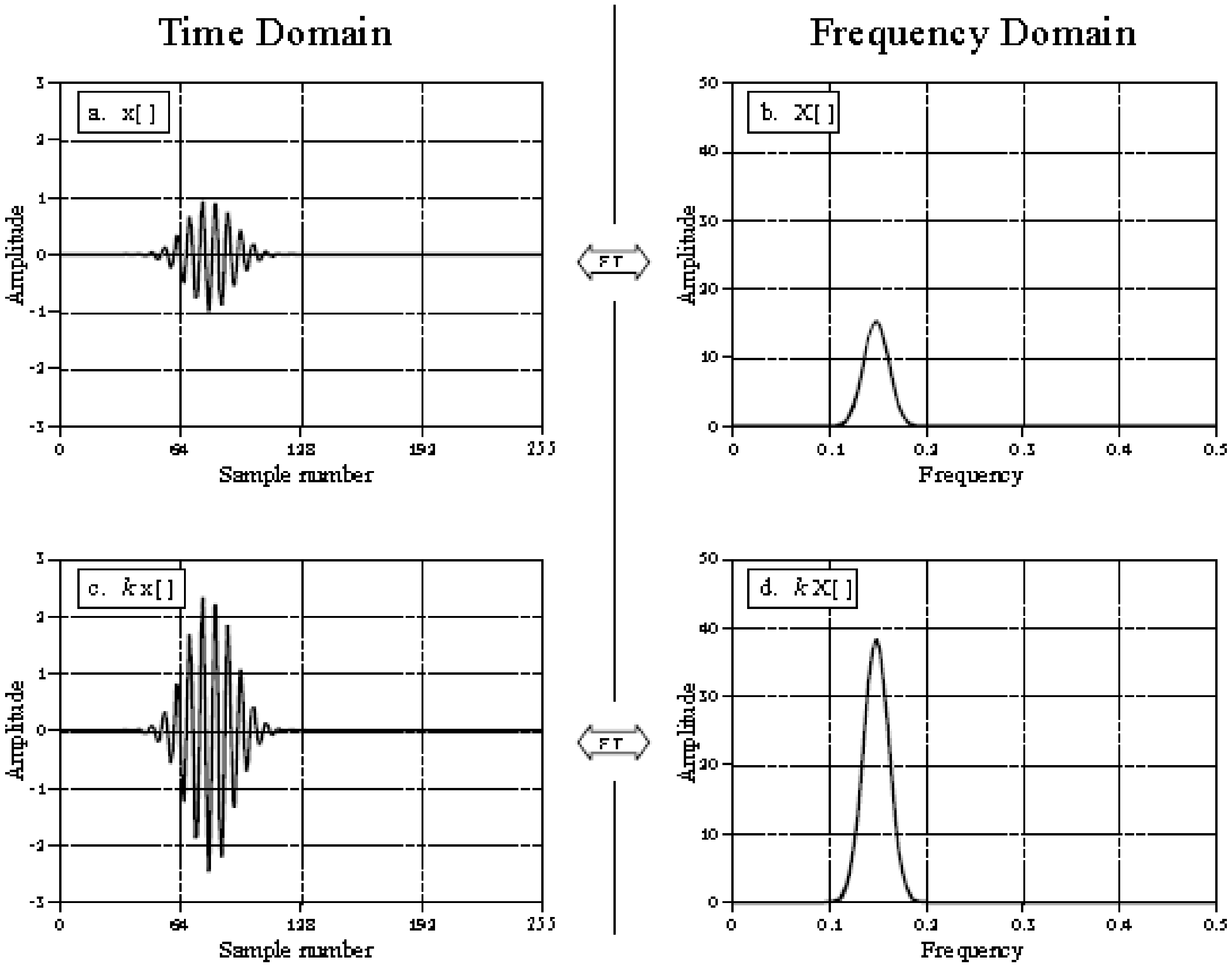


FIGURE 10-1
Homogeneity of the Fourier transform. If the amplitude is changed in one domain, it is changed by the same amount in the other domain. In other words, *scaling* in one domain corresponds to *scaling* in the other domain.

FT is linear:

- **Homogeneity**
- **Additivity**

Homogeneity:

$$x[\,] \xrightarrow{\text{DFT}} X[\,]$$

$$kx[\,] \xrightarrow{\text{DFT}} kX[\,]$$

Frequency is not changed.

Fourier Transform Properties: Additivity

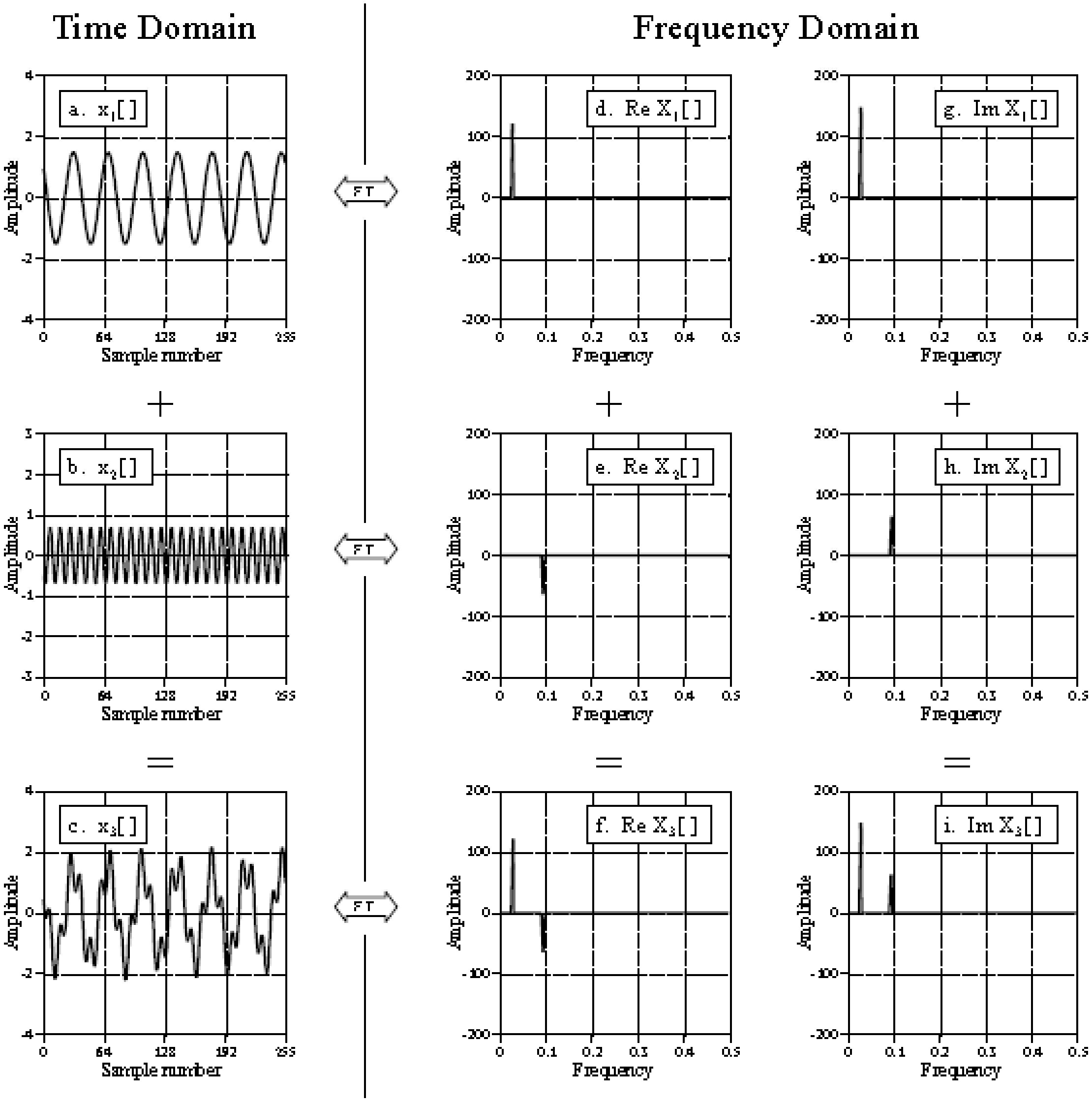


FIGURE 10-2
Additivity of the Fourier transform. Adding two or more signals in one domain results in the corresponding signals being added in the other domain. In this illustration, the time domain signals in (a) and (b) are added to produce the signal in (c). This results in the corresponding real and imaginary parts of the frequency spectra being added.

$$If: x_1[n] + x_2[n] = x_3[n]$$

$$Then: \mathrm{Re}\, X_1[f] + \mathrm{Re}\, X_2[f] = \mathrm{Re}\, X_3[f]$$

$$and \quad \mathrm{Im}\, X_1[f] + \mathrm{Im}\, X_2[f] = \mathrm{Im}\, X_3[f]$$

Information Contained in the Phase

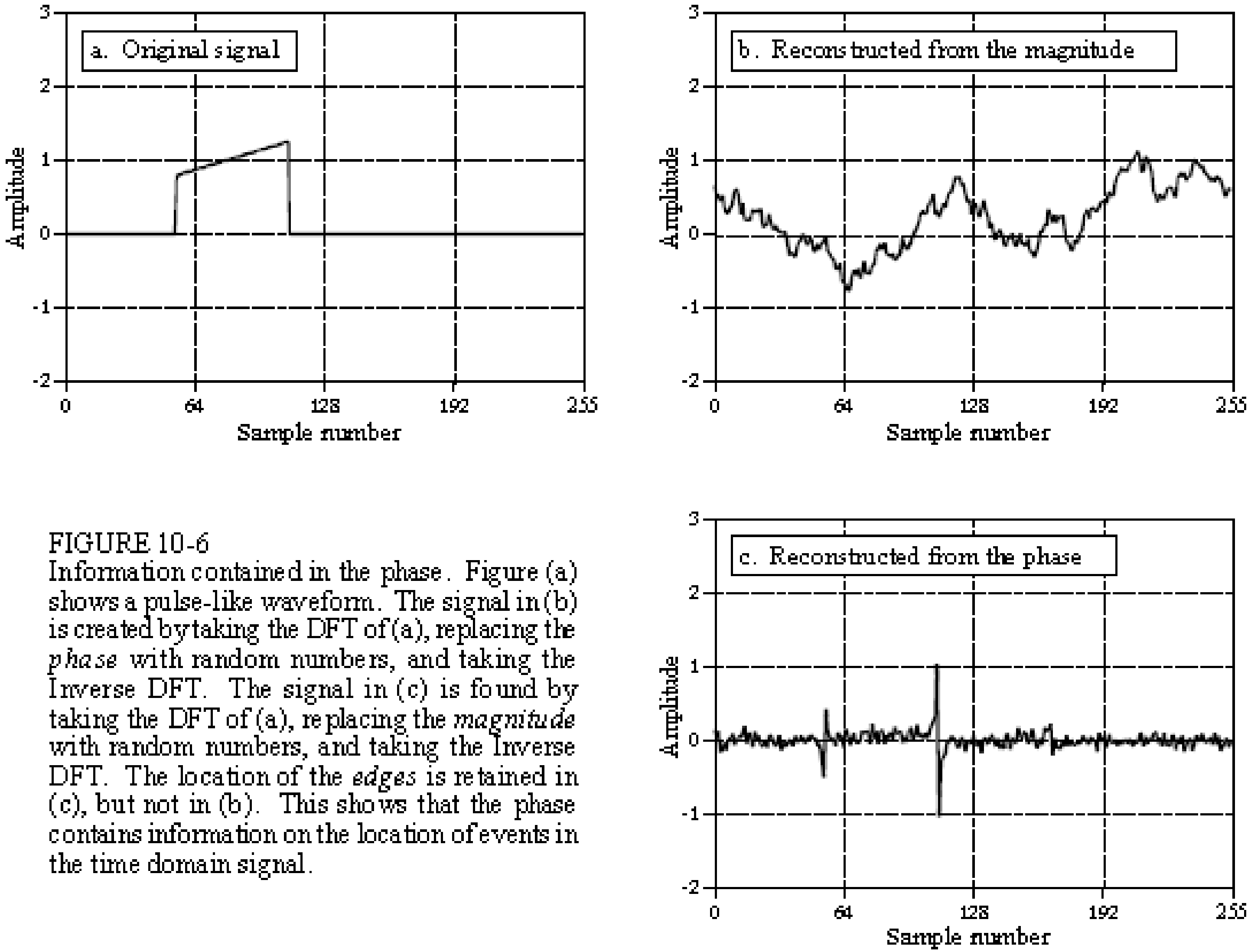


FIGURE 10-6
Information contained in the phase. Figure (a) shows a pulse-like waveform. The signal in (b) is created by taking the DFT of (a), replacing the *phase* with random numbers, and taking the Inverse DFT. The signal in (c) is found by taking the DFT of (a), replacing the *magnitude* with random numbers, and taking the Inverse DFT. The location of the *edges* is retained in (c), but not in (b). This shows that the phase contains information on the location of events in the time domain signal.

Phase Characteristics of Left-Right Symmetry

Left-right flip cancels out non-linearity of phase

Complex Conjugate

Same magnitude

Opposite sign phase

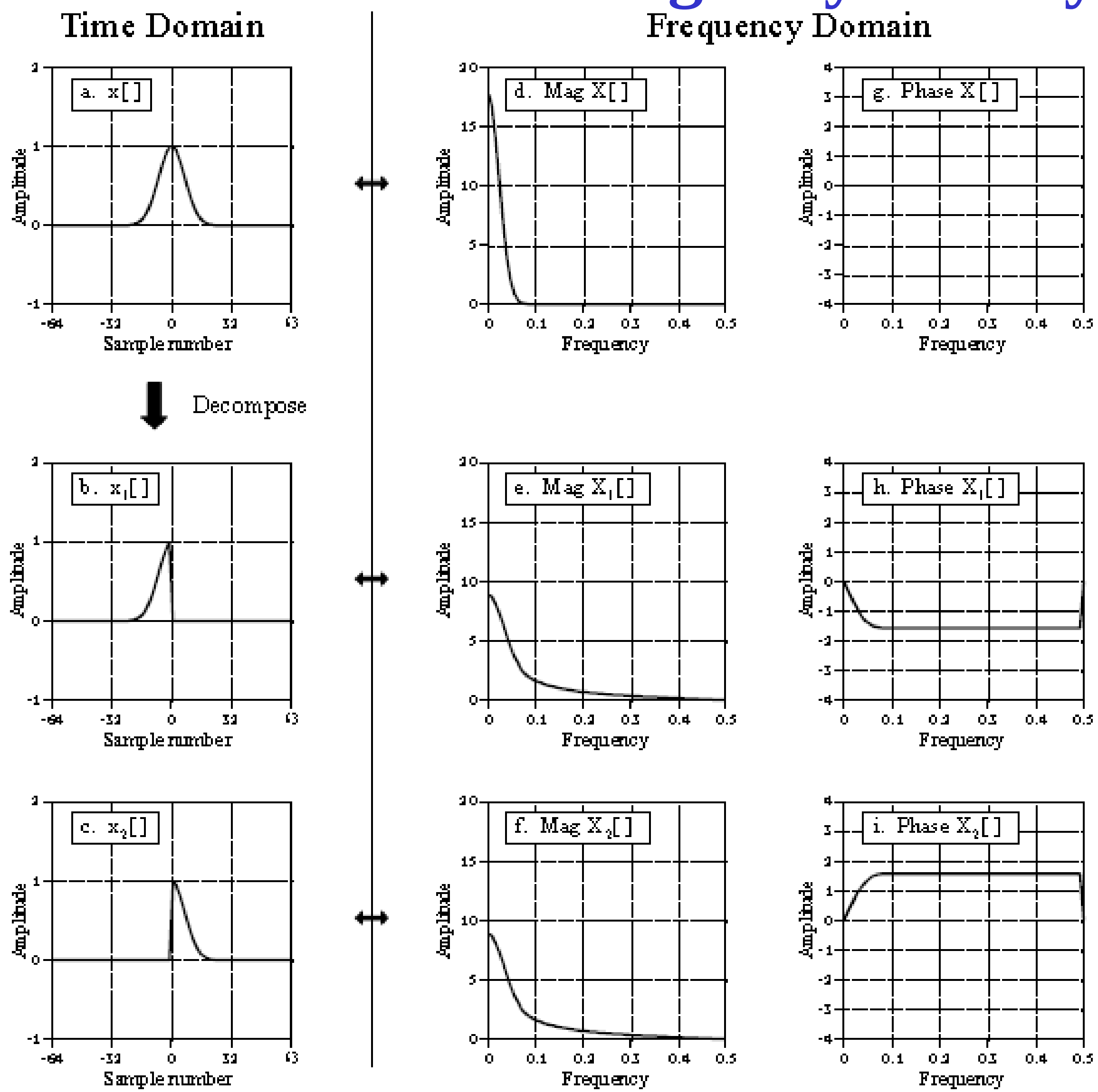


FIGURE 10-7
Phase characteristics of left-right symmetry. A signal with left-right symmetry, shown in (a), can be decomposed into a right half, (b), and a left half, (c). The magnitudes of the two halves are identical, (e) and (f), while the phases are the negative of each other, (h) and (i).

Compression and Expansion

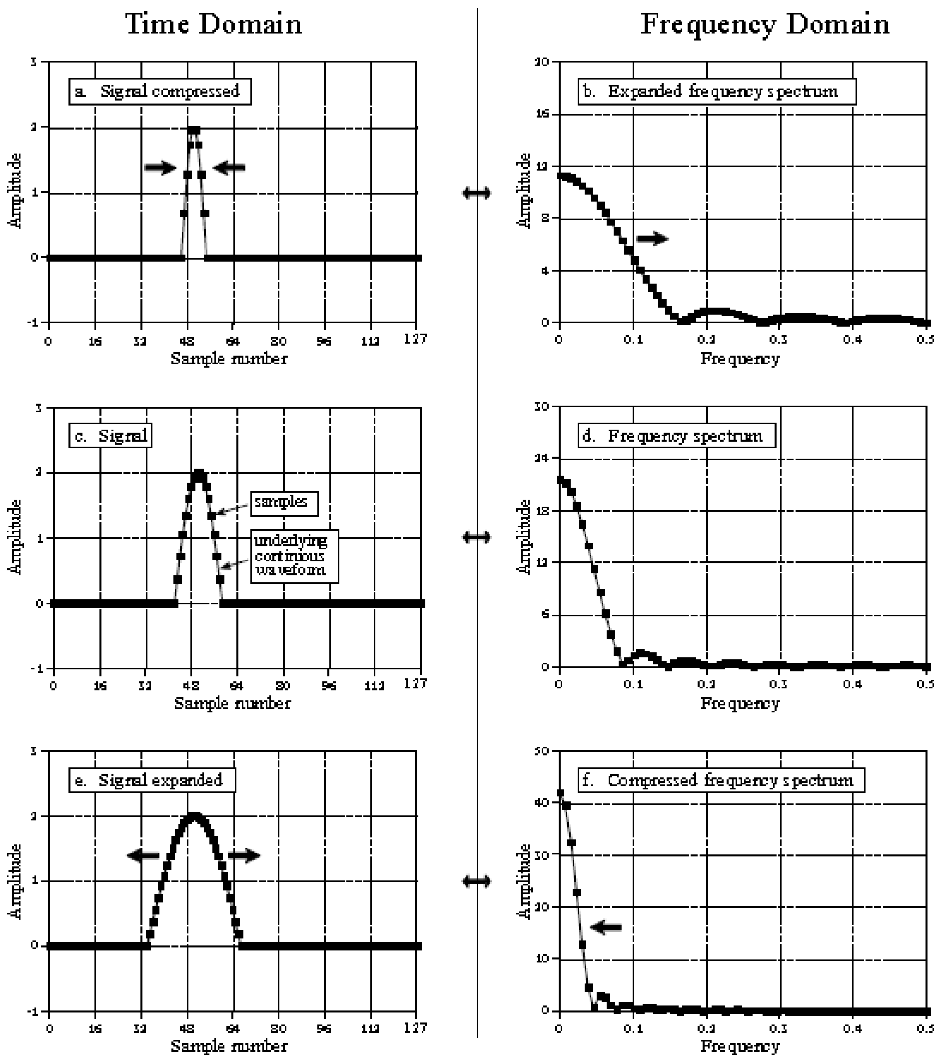


FIGURE 10-12
Compression and expansion. Compressing a signal in one domain results in the signal being expanded in the other domain, and vice versa. Figures (c) and (d) show a discrete signal and its spectrum, respectively. In (a) and (b), the time domain signal has been compressed, resulting in the frequency spectrum being expanded. Figures (e) and (f) show the opposite process. As shown in these figures, discrete signals are expanded or contracted by expanding or contracting the underlying continuous waveform. This underlying waveform is then resampled to find the new discrete signal.

Vice versa in time & frequency domain

<u>Time domain</u>		<u>Frequency domain</u>
Impulse	↔	Constant
Constant	↔	Impulse

Amplitude Modulation

Modulation: Process of merging two signals to form a third signal with desirable characteristics of both.

AM: Multiplying two signals (audio signal and carrier wave) in time domain.

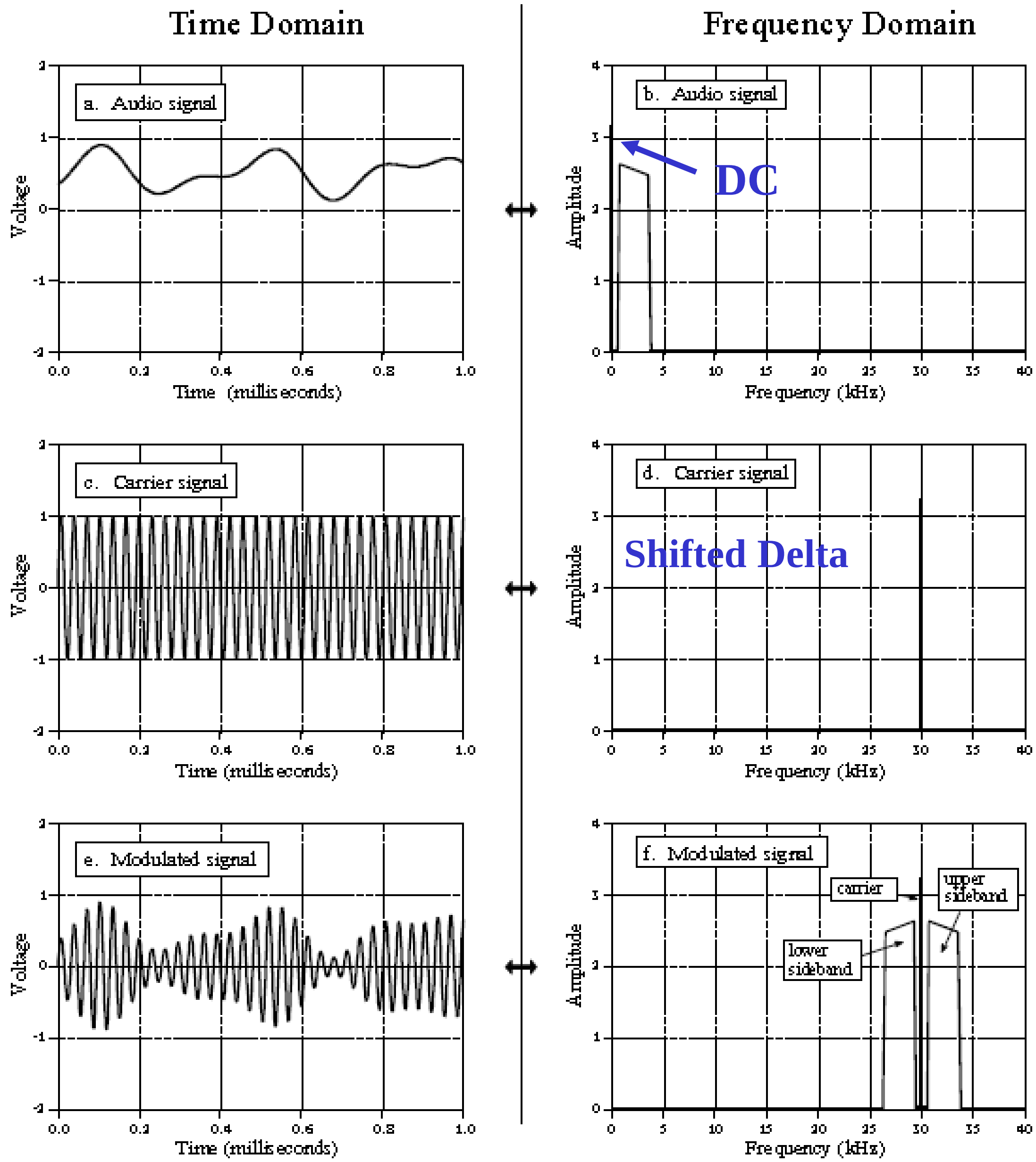


FIGURE 10-14
Amplitude modulation. In the time domain, amplitude modulation is achieved by multiplying the audio signal, (a), by the carrier signal, (c), to produce the modulated signal, (e). Since multiplication in the time domain corresponds to convolution in the frequency domain, the spectrum of the modulated signal is the spectrum of the audio signal shifted to the frequency of the carrier.

Fourier Transform Pairs

Delta Function Pairs in Polar Form

Delta Function

Shifted Delta Function

Same Magnitude, Different Phase

Shifted Delta Function

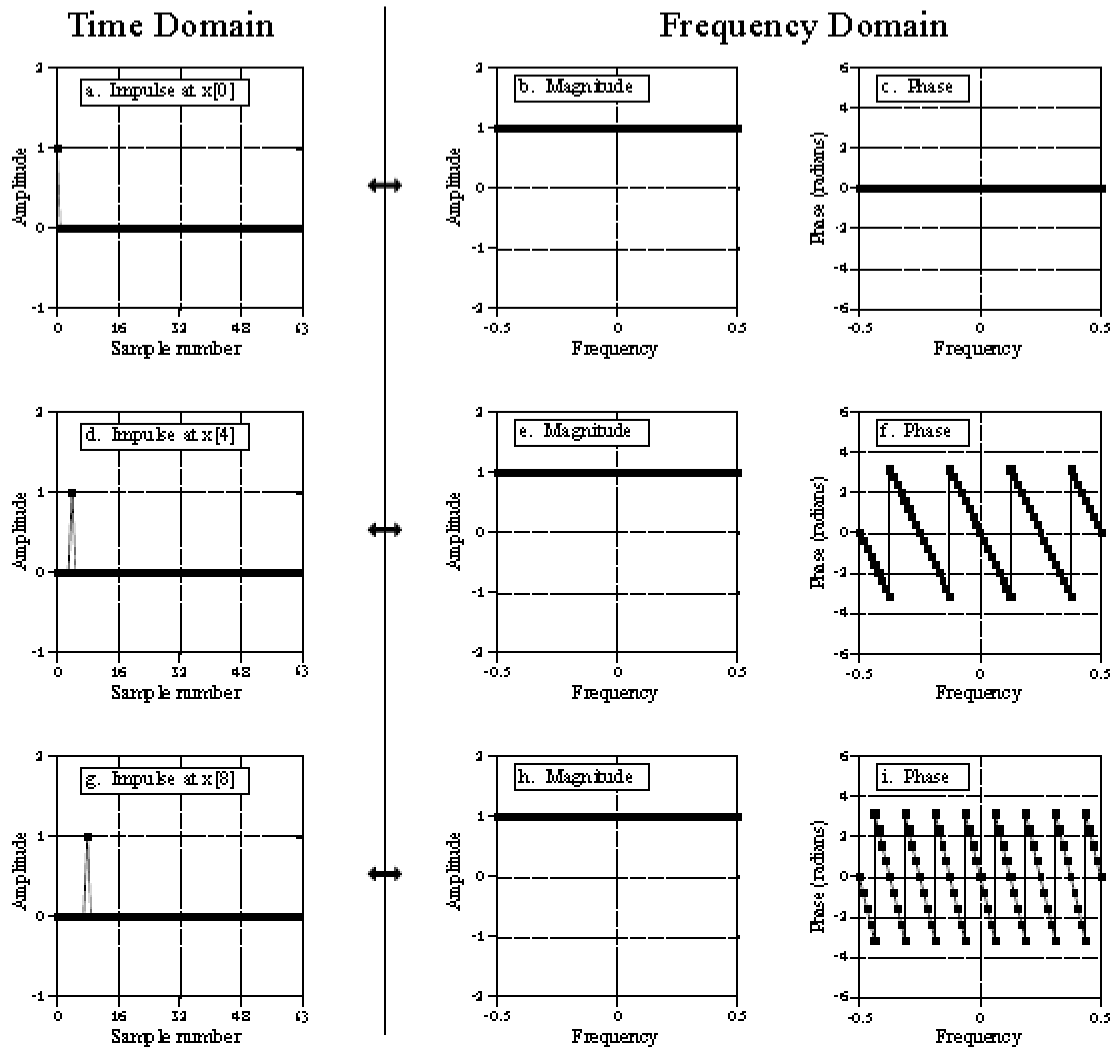


FIGURE 11-1
Delta function pairs in *polar form*. An impulse in the time domain corresponds to a constant magnitude and a linear phase in the frequency domain.

Delta Function Pairs in Rectangular Form

Delta Function

Shifted Delta Function

Shifted Delta Function

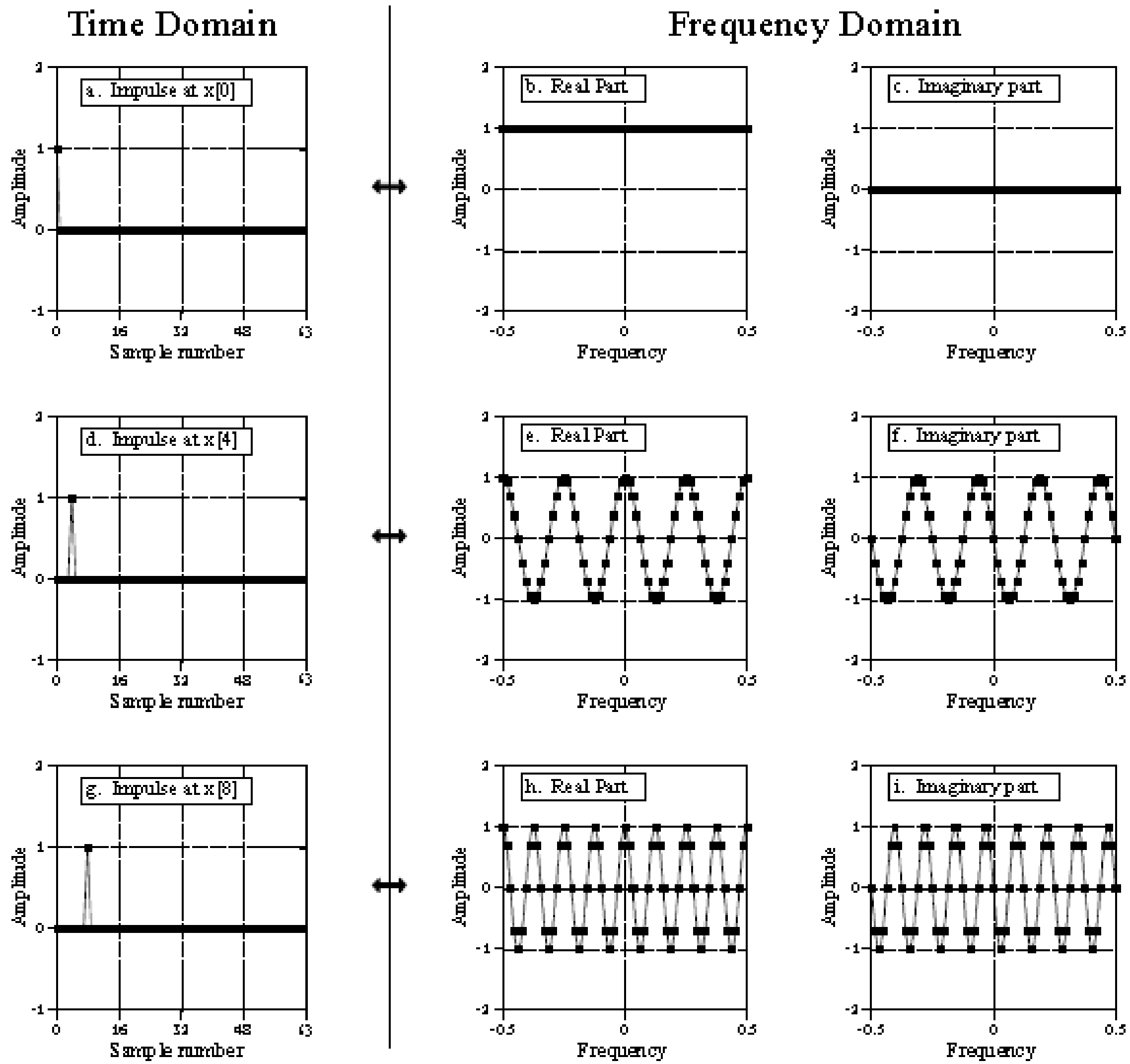


FIGURE 11-2
Delta function pairs in *rectangular form*. Each sample in the time domain results in a cosine wave in the real part, and a negative sine wave in the imaginary part of the frequency domain.

Common Transform Pairs

Filter kernel of low-pass filter

Perfect low-pass filter

Sine wave × Gaussian = Gaussian burst

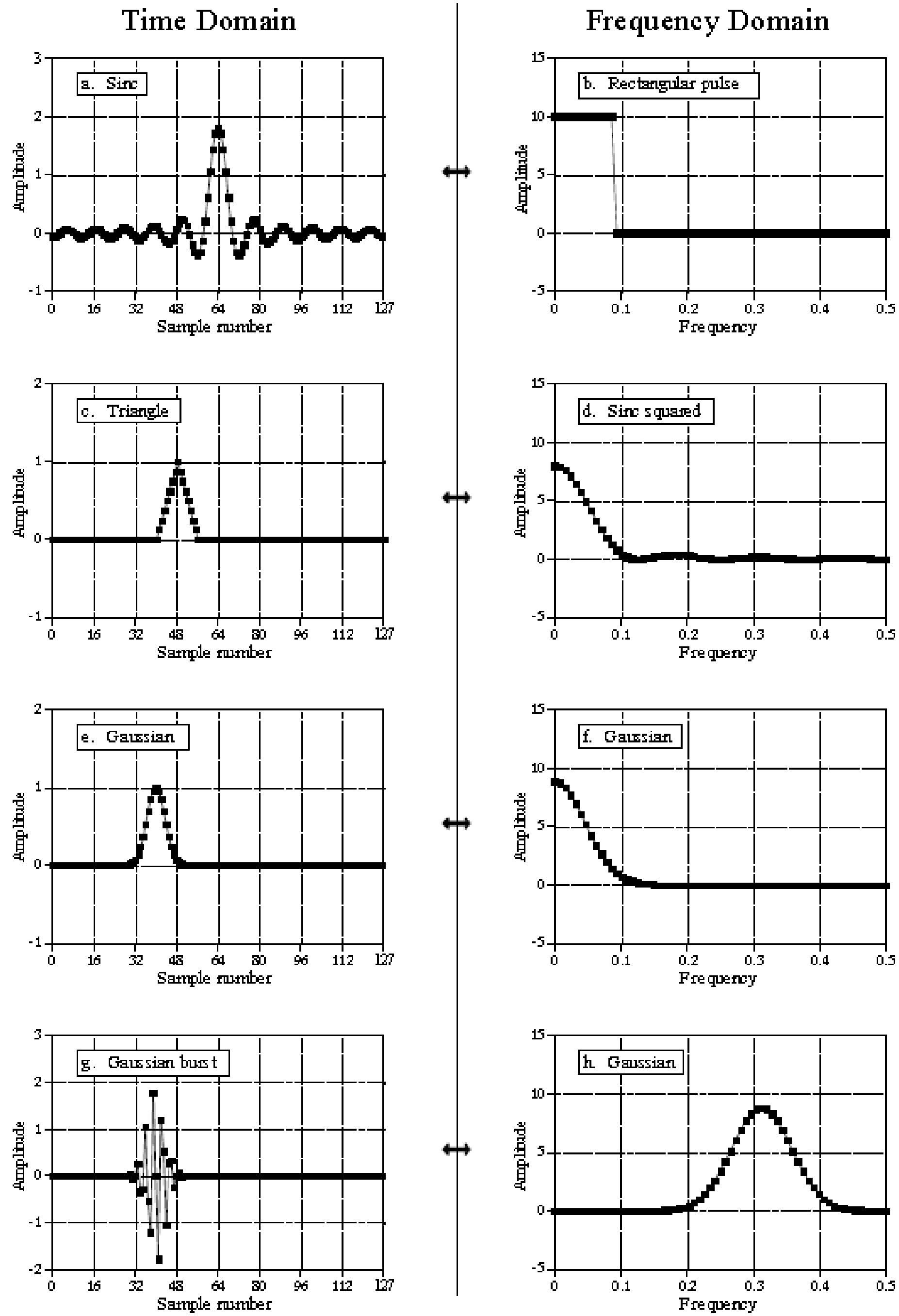


FIGURE 11-5
Common transform pairs.

Gibbs Effect

When only some of the frequency components are used for wave reconstruction, each edge shows overshoot and ringing.

When more sinusoids are added, the width of overshoot decreases, but height remains same

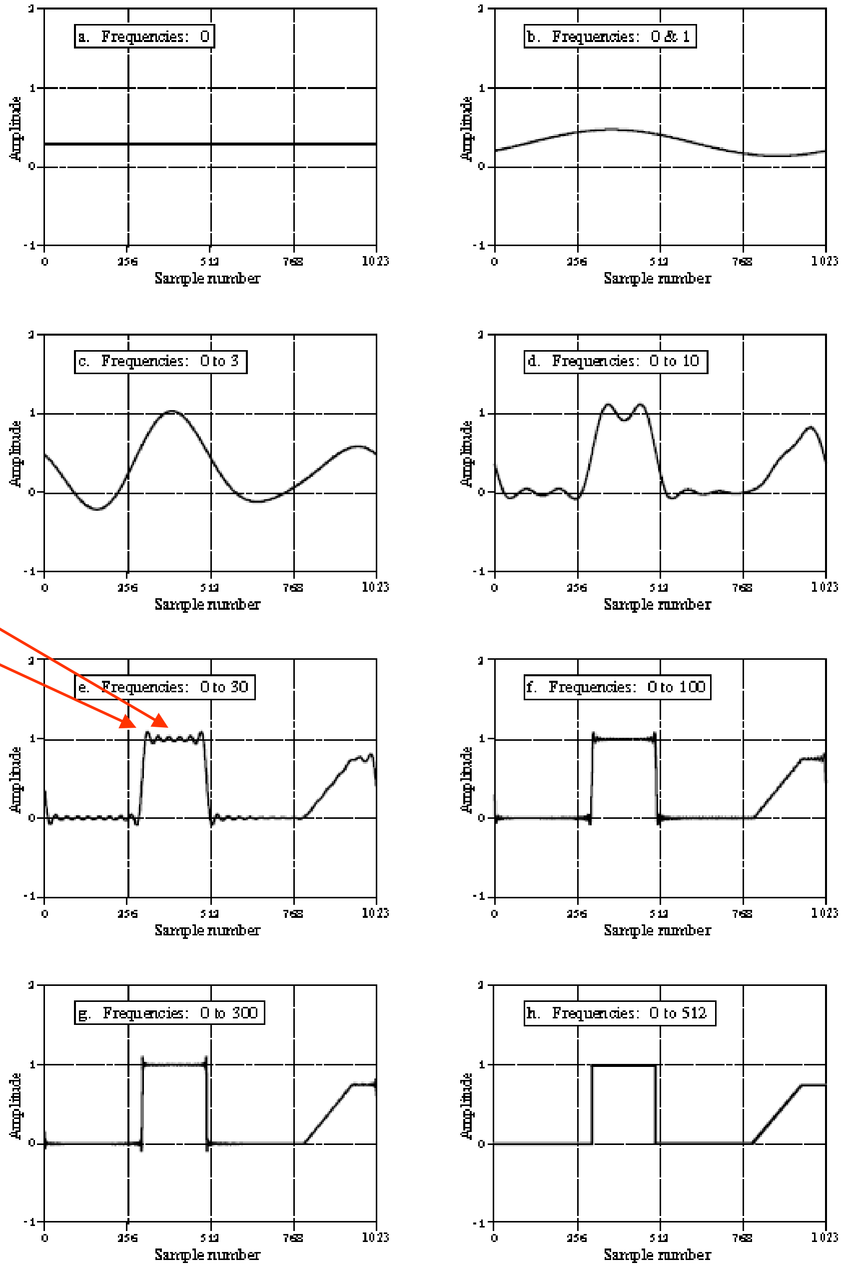


FIGURE 11-6.
The Gibbs effect.

Harmonics

If a signal is periodic with frequency *f*, the only frequencies composing the signal are integer multiple of *f*, i.e., *f*, 2*f*, 3*f*, 4*f*, etc.

Pure sinusoid have only one frequency.

Symmetric distortion cancels out even harmonics.

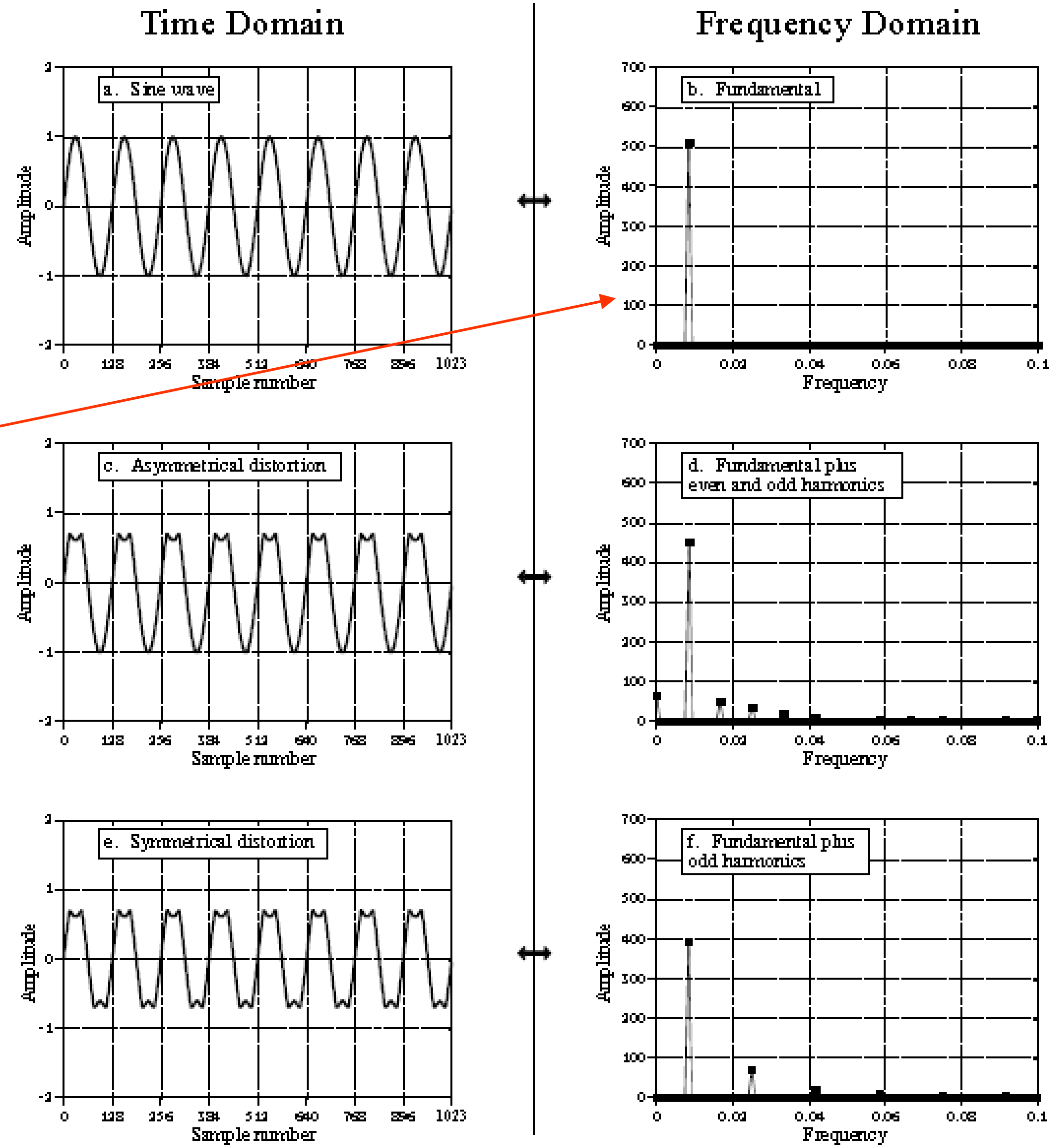


FIGURE 11-7
Example of harmonics. Asymmetrical distortion, shown in (c), results in even and odd harmonics, (d), while symmetrical distortion, shown in (e), produces only odd harmonics, (f).

Fast Fourier Transform (FFT)

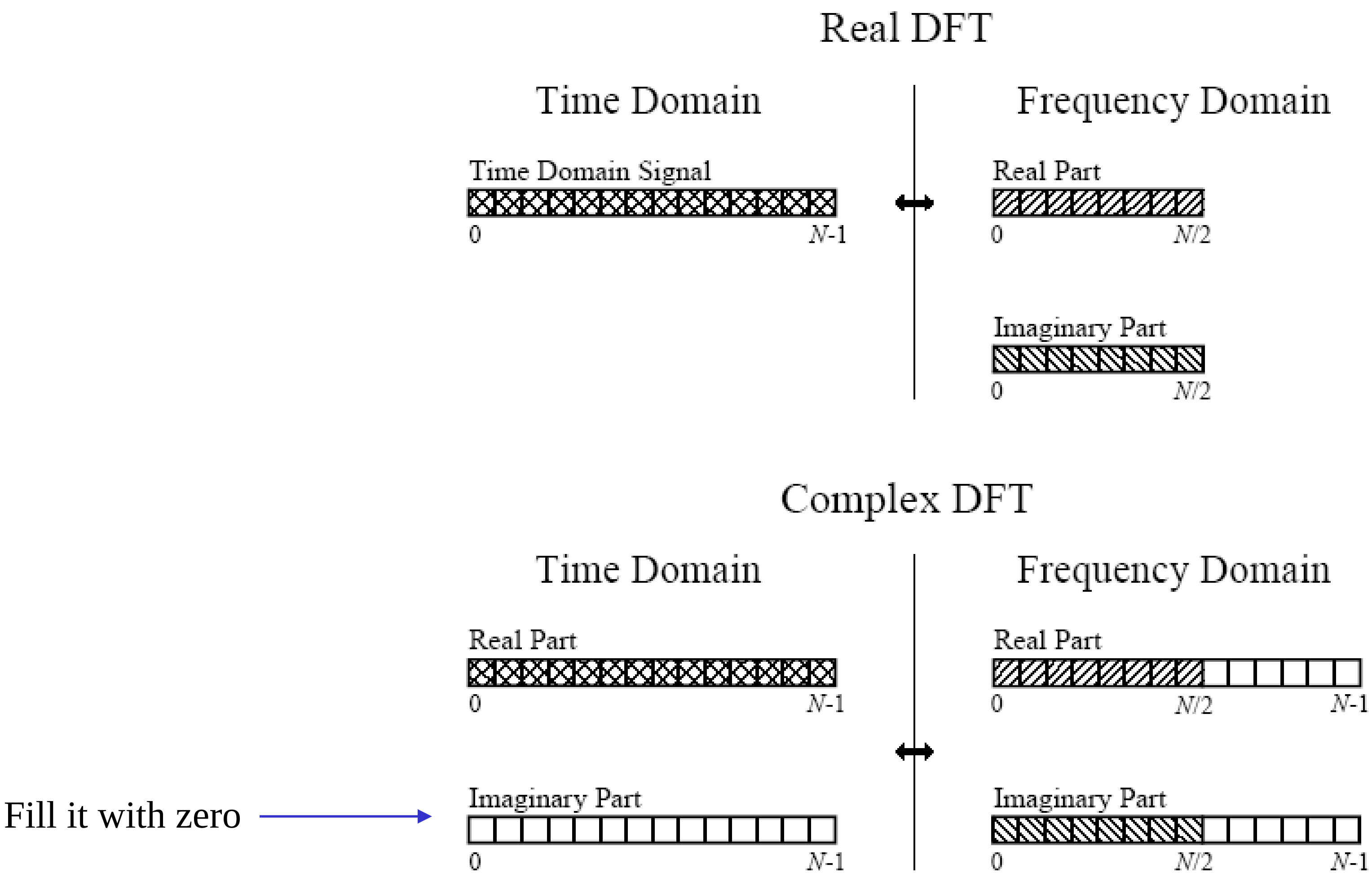


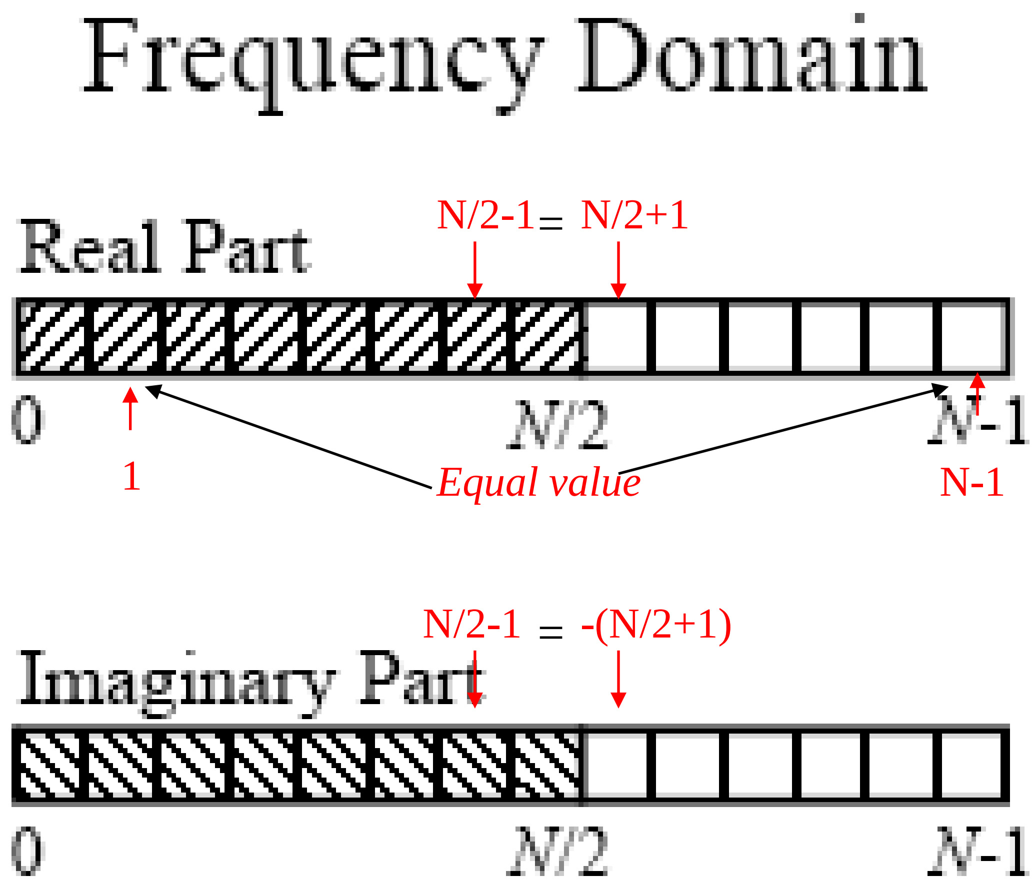
FIGURE 12-1
Comparing the real and complex DFTs. The real DFT takes an N point time domain signal and creates two $N/2 + 1$ point frequency domain signals. The complex DFT takes two N point time domain signals and creates two N point frequency domain signals. The crosshatched regions shows the values common to the two transforms.

Complex DFT

Positive frequencies: from 0 to N/2.

Negative frequencies: from N/2 to N-1.

No matching pairs for samples at 0 and N/2.



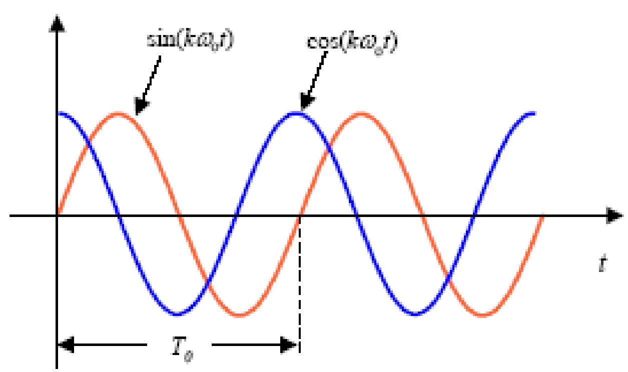
DFT Matrix

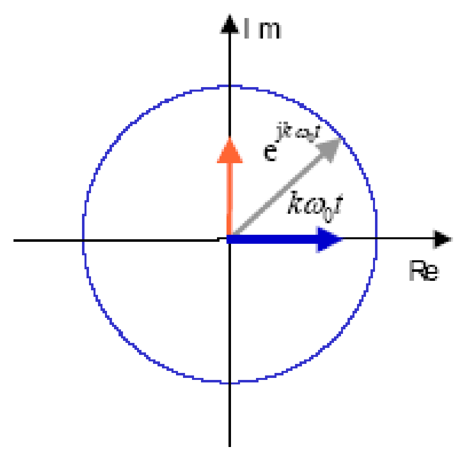
Frequency Spectrum

Multiplication Matrix

Time-Domain samples

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-2) \\ X(N-1) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{4\pi}{N}} & \cdots & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{2(N-1)\pi}{N}} \\
1 & e^{-j\frac{4\pi}{N}} & e^{-j\frac{8\pi}{N}} & \cdots & e^{-j\frac{4(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} & \cdots & e^{-j\frac{2(N-2)^2\pi}{N}} & e^{-j\frac{2(N-2)(N-1)\pi}{N}} \\
1 & e^{-j\frac{2(N-1)\pi}{N}} & e^{-j\frac{4(N-1)\pi}{N}} & \cdots & e^{-j\frac{2(N-1)(N-2)\pi}{N}} & e^{-j\frac{(N-1)^2\pi}{N}}
\end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix}
$$





DFT Matrix

Continued

Let, $w_N = e^{-j2\pi/N}$

Then

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{(N-1)} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(N-1)} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

DFT equation: $$X(k) = \sum_{m=0}^{N-1} x(m) w_N^{mk} \qquad k = 0, \ldots, N-1$$

DFT to FFT (1)

Decimation-in-frequency

DFT: $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$ for $k = 0, 1, \ldots, N-1,$

$$X(k) = x(0) + x(1) W_N^k + \ldots + x(N-1) W_N^{k(N-1)}$$

$$X(k) = \boxed{x(0) + x(1) W_N^k + \ldots + x\left(\frac{N}{2}-1\right) W_N^{k(N/2-1)}} + \boxed{x\left(\frac{N}{2}\right) W^{kN/2} + \ldots + x(N-1) W_N^{k(N-1)}}$$

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \boxed{\sum_{n=N/2}^{N-1} x(n) W_N^{kn}}$$

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{kn}.$$

$$W_N^{N/2} = e^{-j\frac{2\pi(N/2)}{N}} = e^{-j\pi} = -1;$$

$$X(k) = \sum_{n=0}^{(N/2)-1} \left(x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right) W_N^{kn}$$

Now decompose into even (k = 2m) and odd (k = 2m+1) sequences.

DFT to FFT (2)

$$X(2m) = \sum_{n=0}^{(N/2)-1} \left(x(n) + x\left(n + \frac{N}{2}\right)\right) W_N^{2mn}$$

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} \left(x(n) - x\left(n + \frac{N}{2}\right)\right) W_N^n W_N^{2mn}$$

$$W_N^2 = e^{-j\frac{2\pi \times 2}{N}} = e^{-j\frac{2\pi}{(N/2)}} = W_{N/2}$$

$$X(2m) = \sum_{n=0}^{(N/2)-1} a(n) W_{N/2}^{mn} = DFT\{a(n) \text{ with } (N/2) \text{ points}\}$$

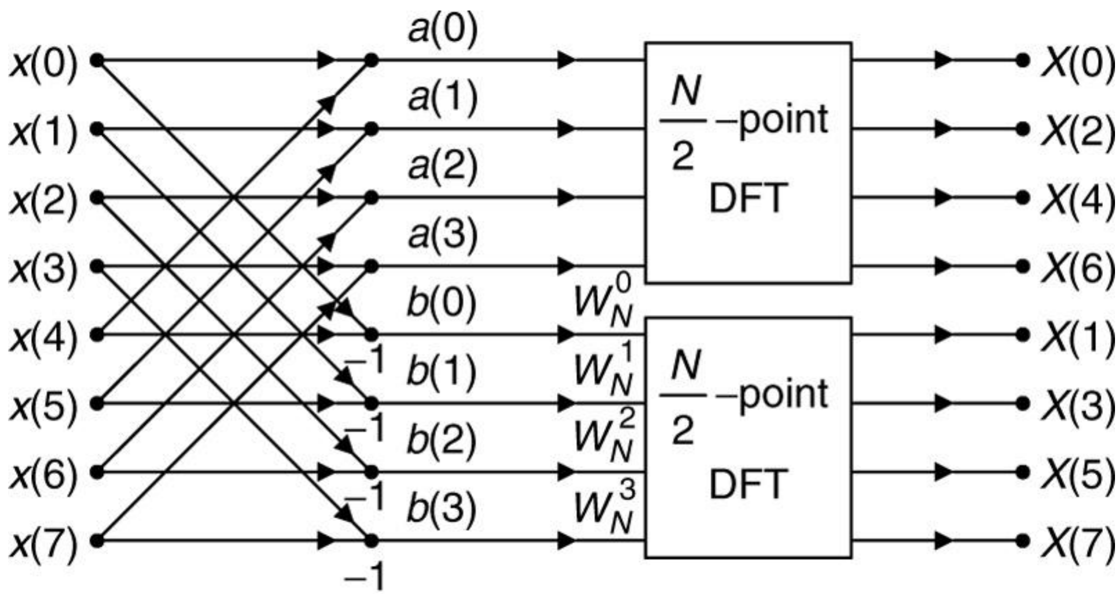
$$X(2m+1) = \sum_{n=0}^{(N/2)-1} b(n) W_N^n W_{N/2}^{mn} = DFT\{b(n) W_N^n \text{ with } (N/2) \text{ points}\}$$

$$a(n) = x(n) + x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1 \ldots, \frac{N}{2} - 1$$

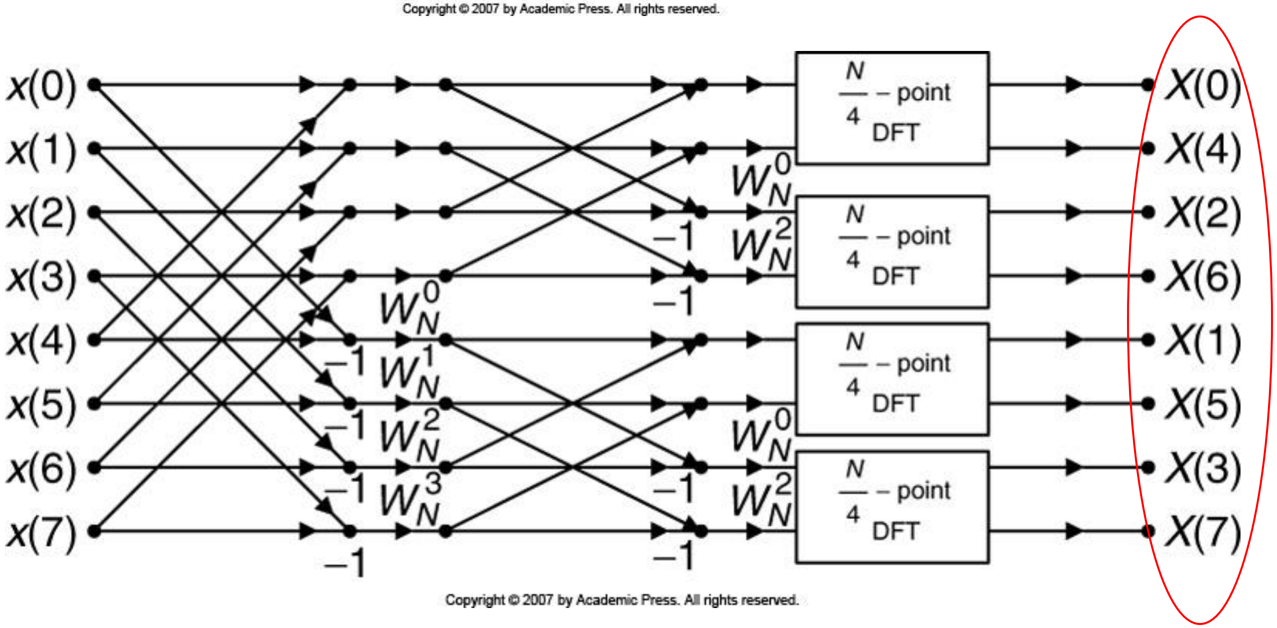
$$b(n) = x(n) - x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1, \ldots, \frac{N}{2} - 1.$$

$$DFT\{x(n) \text{ with } N \text{ points}\} = \begin{cases} DFT\{a(n) \text{ with } (N/2) \text{ points}\} \\ DFT\{b(n) W_N^n \text{ with } (N/2) \text{ points}\} \end{cases}$$

DFT to FFT (3)

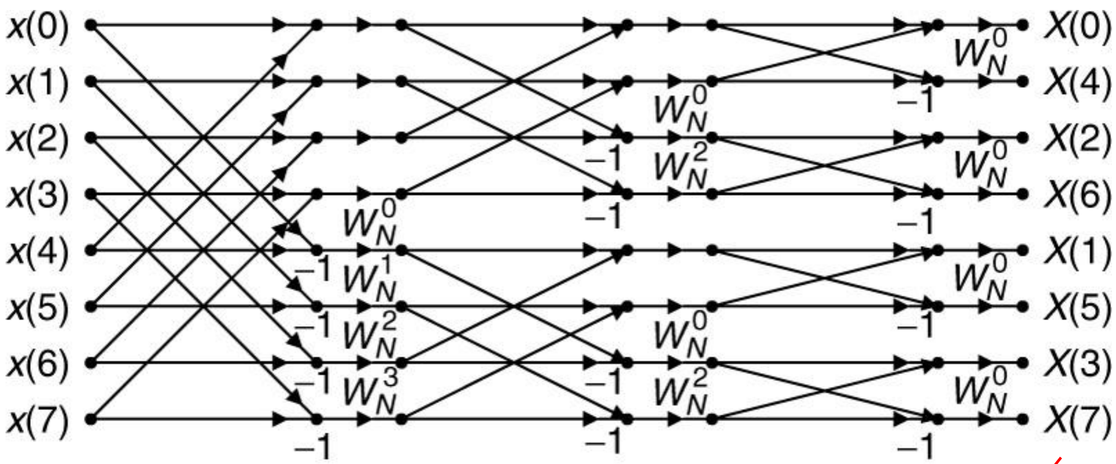


x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
a(0)
a(1)
a(2)
a(3)
b(0)
b(1)
b(2)
b(3)
−1
−1
−1
−1
W_N^0
W_N^1
W_N^2
W_N^3
$\frac{N}{2}$ –point DFT
$\frac{N}{2}$ –point DFT
X(0)
X(2)
X(4)
X(6)
X(1)
X(3)
X(5)
X(7)



x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
W_N^0
W_N^1
W_N^2
W_N^3
−1
−1
−1
−1
W_N^0
W_N^2
W_N^0
W_N^2
$\frac{N}{4}$ – point DFT
$\frac{N}{4}$ – point DFT
$\frac{N}{4}$ – point DFT
$\frac{N}{4}$ – point DFT
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

DFT to FFT (4)



12 complex multiplication

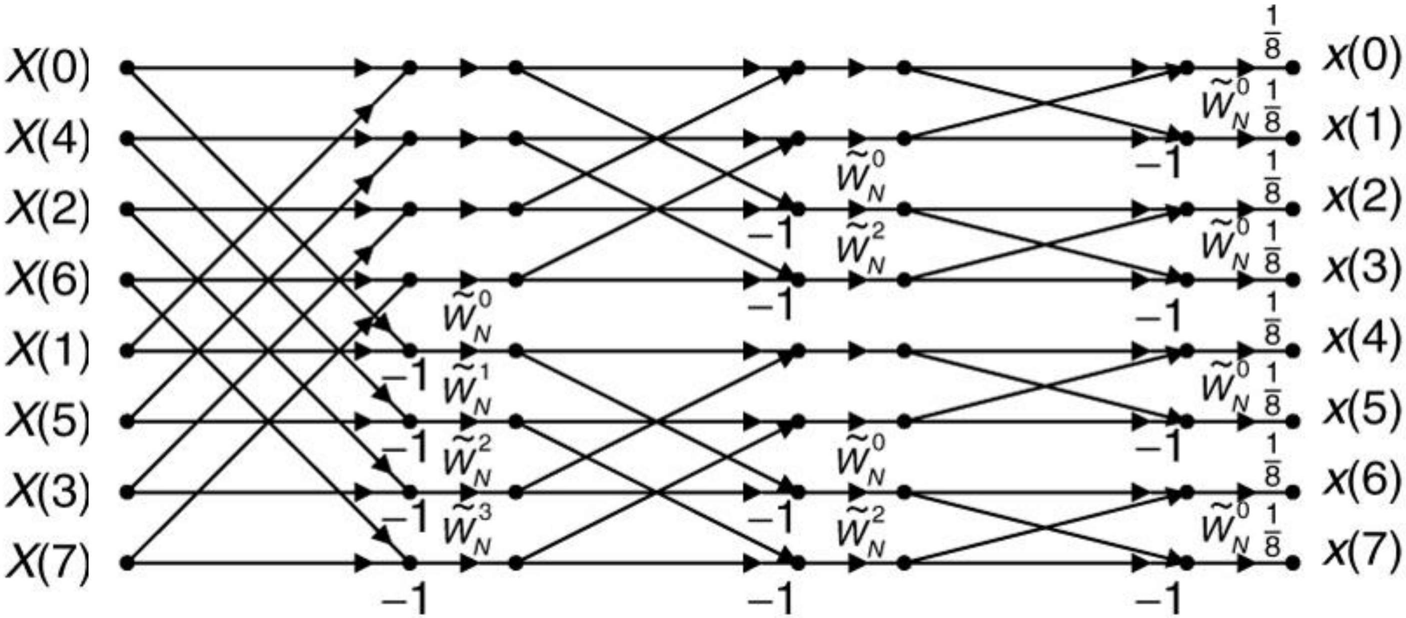
Binary	index	1st split	2nd split	3rd split	Bit reversal
000	0	0	0	0	000
001	1	2	4	4	100
010	2	4	2	2	010
011	3	6	6	6	011
100	4	1	1	1	001
101	5	3	5	5	101
110	6	5	3	3	011
111	7	7	7	7	111

Complex multiplications of DFT $= N^2$, and

Complex multiplications of FFT $= \frac{N}{2}\log_2(N)$

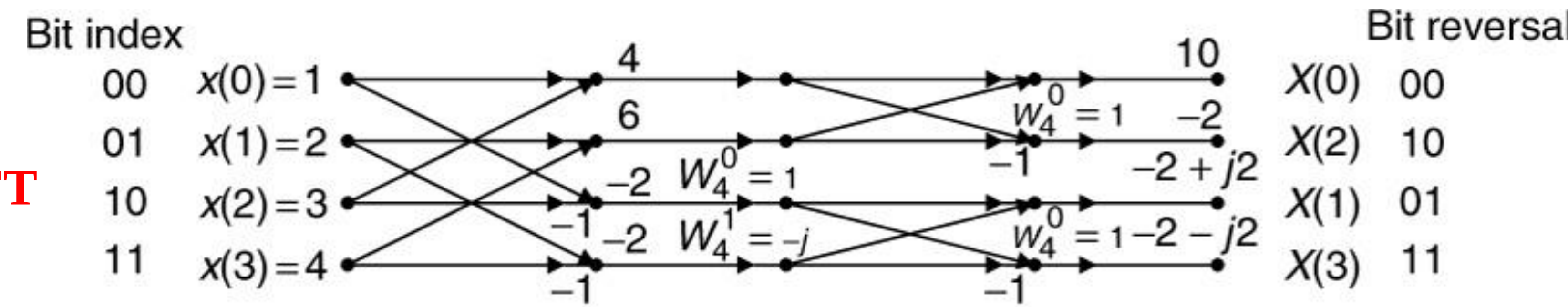
IFFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\tilde{W}_N^{kn}, \text{ for } k = 0, 1, \ldots, N-1.$$



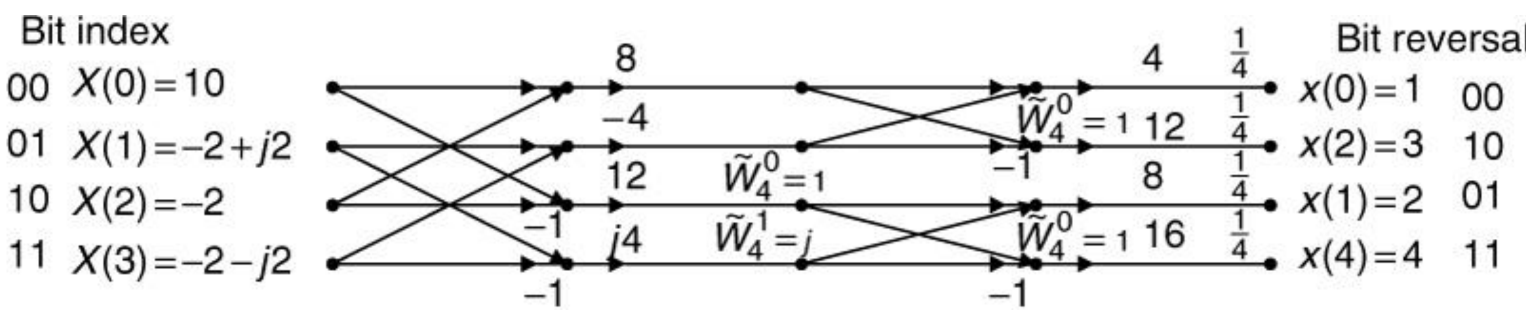
Example: FFT & IFFT

FFT



Number of complex multiplication = $\frac{N}{2}\log_2(N) = \frac{4}{2}\log_2(4) = 4.$

IFFT



DFT to FFT

Dividing the input sequence into *even* and *odd* sequences:

$$X(k) = \sum_{m \in even} x(m) w_N^{mk} + \sum_{m \in odd} x(m) w_N^{mk}$$

$$X(k) = \sum_{m=0}^{N/2-1} x(2m) w_N^{2mk} + \sum_{m=0}^{N/2-1} x(2m+1) w_N^{(2m+1)k}$$

$$X(k) = \sum_{m=0}^{N/2-1} x(2m) \left(w_N^2\right)^{mk} + w_N^k \sum_{m=0}^{N/2-1} x(2m+1) \left(w_N^2\right)^{mk}$$

Now $w_N = e^{-j2\pi/N}$ hence

$$w_N^2 = \left(e^{-j2\pi/N}\right)^2 = e^{-j2\pi/(N/2)} = w_{N/2}$$

$$X(k) = \sum_{m=0}^{N/2-1} x(2m) w_{N/2}^{mk} + w_N^k \sum_{m=0}^{N/2-1} x(2m+1) w_{N/2}^{mk}$$

N-point DFT is composed of 2 N/2-point DFTs.

Complexity is reduced by half from N(N-1) to 2(N/2)(N/2)

This can be iterated $\log_2 N-1 = \log_2(N/2)$ times to reach DFT of 2-points.

FFT: decimation-in-time

Decompose a signal of N points into N signals of 1 point.

Log_2N stages required

$16 = 2^4$; 4 stages req.

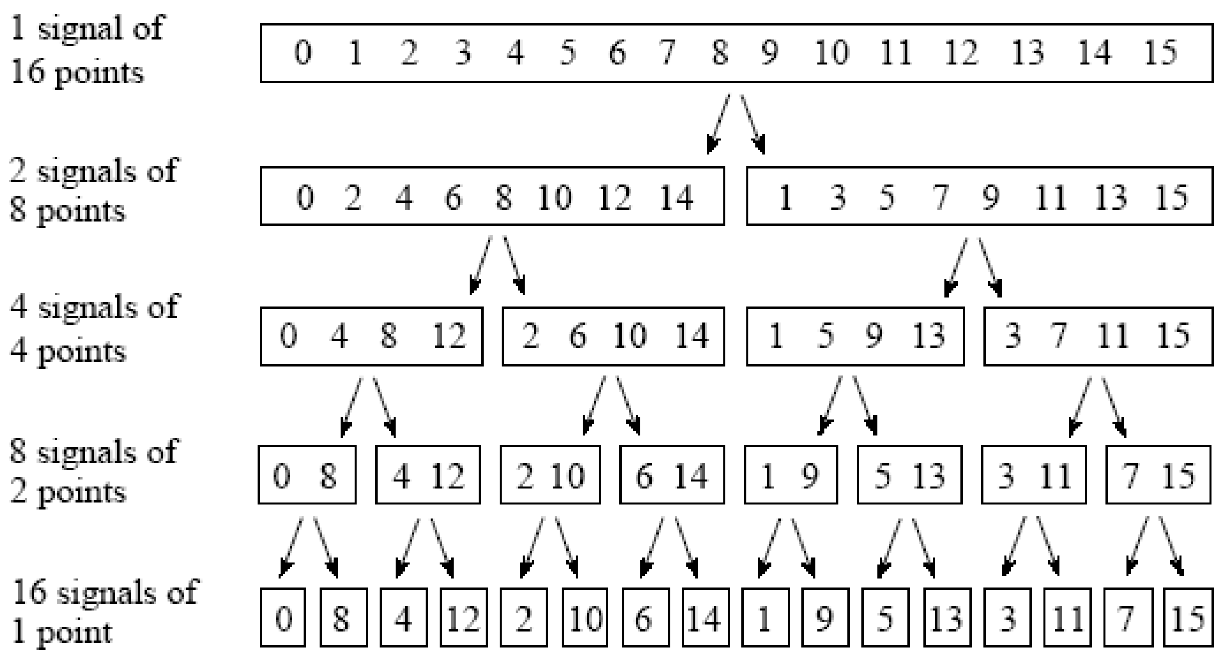
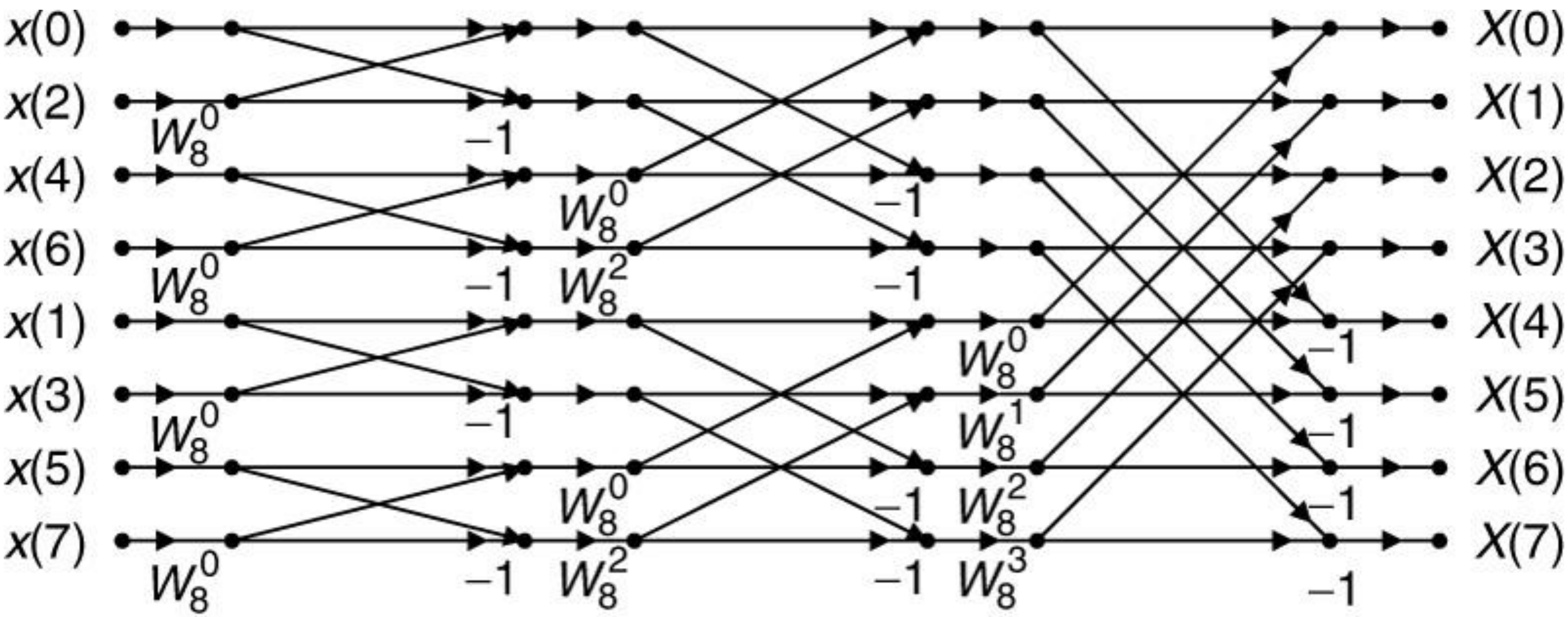


FIGURE 12-2
The FFT decomposition. An *N* point signal is decomposed into *N* signals each containing a single point. Each stage uses an *interlace decomposition*, separating the even and odd numbered samples.

8-Point FFT

Decimation-in-time



$$W_N = e^{-\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

$$W_8^2 = e^{-\frac{2\pi\times 2}{8}} = e^{-\frac{\pi}{2}} = \cos(\pi/2) - j\sin(\pi/2) = -j$$