Modeling in The Time Domain

State-space Method

Frequency domain approach (classical approach):

based on converting a system's differential equation to a transfer function.

Advantage: rapidly providing stability and transient response information. Thus we can immediately see the effects of varying system parameters.

Disadvantage: limited application. It can be applied only to linear, time-invariant systems or systems that can be approximated as such.

State-space approach (time domain / modern approach):

Can be used: a) To represent non-linear systems that have backlash, saturation, dead zone.

b) It can handle systems with nonzero initial conditions.

c) Multiple-inputs, multiple-outputs systems can easily be represented.

d) Many commercial software packages are available.

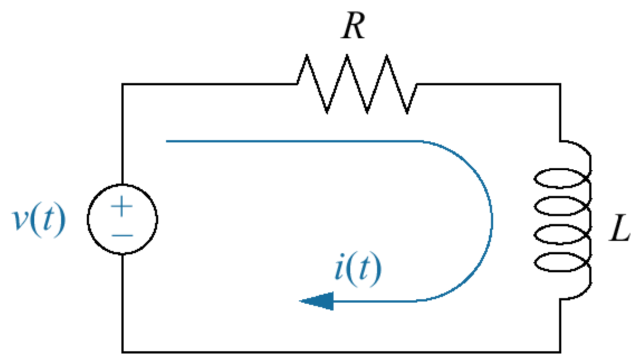
Many calculation is needed before actual realization.

RL Network: State-Space Representation

1. Select a state variable: say $i(t)$.

2. Write differential equation (of $i(t)$):

$$L\frac{di}{dt}+Ri=v(t)$$



3. Take Laplace transform:

$$L[sI(s)-i(0)]+RI(s)=V(s)$$

$$I(s)=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+\frac{R}{L}}\right)+\frac{i(0)}{s+\frac{R}{L}}$$

If $v(t) = u(t)$, then $V(s) = 1/s$.

Inverse Laplace transform:

$$i(t)=\frac{1}{R}\left(1-e^{-(R/L)t}\right)+i(0)e^{-(R/L)t}$$

4. Output equations:

$$v_R(t)=Ri(t)$$

$$v_L(t)=v(t)-Ri(t)$$

$$\frac{di}{dt}=\frac{1}{L}[v(t)-Ri(t)]$$

If we know initial condition of i, $i(0)$, and input voltage, $v(t)$, then we can find the value of any network variables at time $t \geq t_0$.

Self Study

State-space representation of RLC network.

Some Terminology

Linear combination: (of n variables): $S = K_n x_n + K_{n-1} x_{n-1} + \ldots\ldots + K_1 x_1$

Linear independence: S is zero if every K is zero and no x is zero: variables x are linearly independent.

System variable: Any variable that responds to an input or initial conditions in a system.

State variable: The smallest set of linearly independent system variables that completely determines (knowing the value at t_0) the value of system variables for $t \geq t_0$

State vector: A vector whose elements are state variables.

State space: The n-dimensional space whose axes are the state variables.

Some Terminology-continued

State equations: A set of *n* simultaneous, first-order differential equations with *n* variables (state variables).

Output equations: The equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

State-space Representation:

State equation ⟶ $\bar{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ For $t \geq t_0$ and initial conditions, $x(t_0)$.

Output equations ⟶ $\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$

$\mathbf{x}$ = state vector

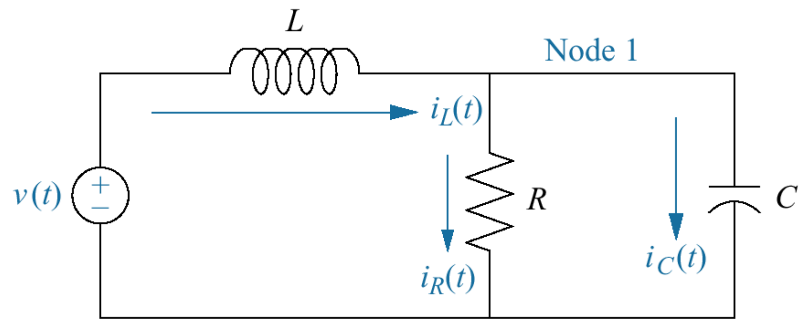
$\bar{\mathbf{x}}$ = derivative of the state vector w.r.t time

$\mathbf{y}$ = output vector

$\mathbf{u}$ = input or control vector

Example-1: State-space Representation

Problem: Find a state-state representation of the following electrical network if the output is the current through the resistor.



Solution:

Step 1: Label all the branch currents in the network.

Step 2: Select the state variables (energy-storage elements: *L* and *C*) and write derivative equations.

$$C\frac{dv_C}{dt} = i_C, \qquad L\frac{di_L}{dt} = v_L$$

Step 3: Express non-state variables (right-hand side: i_C and v_L) as a linear combinations of the state variables (differentiated variables: v_C and i_L) and the input, $v(t)$.

$$i_C = -i_R + i_L = -\frac{1}{R}v_C + i_L$$

$$v_L = -v_C + v(t)$$

Example-1: State-space Representation$_{\text{-contd.}}$

Step 4: Obtain state equations: (by substituting the values and rearranging)

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L, \qquad L\frac{di_L}{dt} = -v_C + v(t)$$

$$\Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L}v_C \qquad + \frac{1}{L}v(t)$$

Step 5: Find the output equation:

$$i_R = \frac{1}{R}v_C$$

Final result: Convert into vector-matrix form
$$\begin{bmatrix} \bar{v}_C \\ \bar{i}_L \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{RC} & \dfrac{1}{C} \\ -\dfrac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} \dfrac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Example-2: State-space Representation

(with a dependent source)

Step 1: Label all the branch currents in the network.

Step 2: Select the state variables (energy-storage elements: *L* and *C*) and write derivative equations.

$$C\frac{dv_C}{dt} = i_C, \qquad L\frac{di_L}{dt} = v_L \qquad x_1 = i_L; \quad x_2 = v_C;$$

Step 3: State equations

$$v_L = v_C + v_{R2} = v_C + i_{R2}R_2$$

At node 2, $i_{R2} = i_C + 4v_L$, so we get,

$$v_L = v_C + (i_C + 4v_L)R_2$$

$$\Rightarrow v_L = \frac{1}{1-4R_2}(v_C + i_C R_2)$$

$$i_C = i(t) - i_{R1} - i_L$$

$$= i(t) - \frac{v_{R1}}{R_1} - i_L$$

$$= i(t) - \frac{v_L}{R_1} - i_L$$

Now solve for v_L and i_C; and rearrange the equations for i_L and v_C.

Step 4: Output equations $v_{R2} = -v_C + v_L;$ $i_{R2} = i_C + 4v_L;$

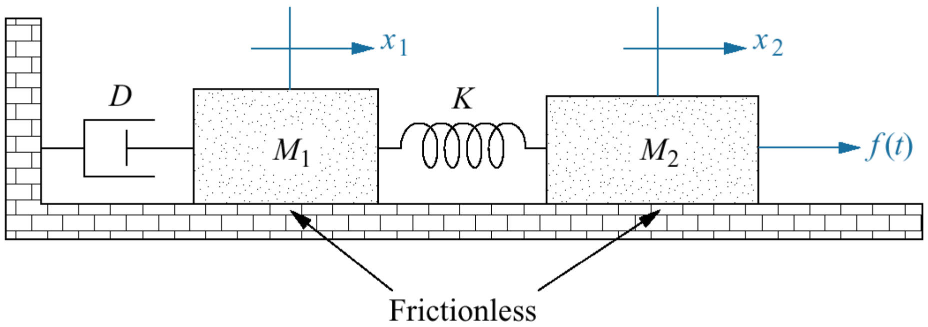
Example-3: State-space Representation

(Translational Mechanical System)

For M_1:

$$M_1 s^2 X_1(s) + DsX_1(s) + KX_1(s) - KX_2(s) = 0$$

$$\Rightarrow M_1 \frac{d^2 x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$



For M_2:

$$-KX_1(s) + KX_2(s) + M_2 s^2 X_2(s) = F(s)$$

$$\Rightarrow -Kx_1 + Kx_2 + M_2 \frac{d^2 x_2}{dt^2} = f(t)$$

Let, $\frac{d^2 x_i}{dt^2} = \frac{dv_i}{dt}$

(acceleration = derivative of velocity)

Select x_1, x_2, v_1, v_2 as state variables.

State equations:

$$\frac{dx_1}{dt} = \qquad + v_1$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

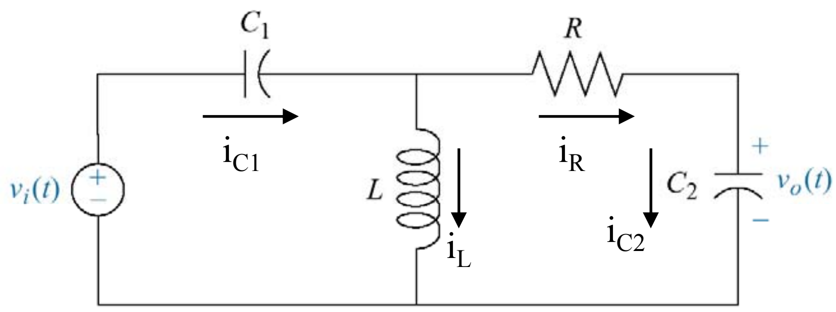
$$\frac{dx_2}{dt} = \qquad + v_2$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 \qquad - \frac{K}{M_2}x_2 \qquad + \frac{1}{M_2}f(t)$$

Write in matrix form.

Example-4: State-space Representation

Problem: Find the state-space representation of the electrical network shown in the figure. The output is $v_0(t)$.



Solution: The derivative relations (one for each energy-storage element)

$$C_1 \frac{dv_{C1}}{dt} = i_{C1}$$

$$L \frac{di_L}{dt} = v_L$$

$$C_2 \frac{dv_{C2}}{dt} = i_{C2}$$

Using Kirchhoff's current and voltage laws:

$$i_{C1} = i_L + i_R = i_L + \frac{1}{R}(v_L - v_{C2})$$

$$v_L = -v_{C1} + v_i$$

$$i_{C2} = i_R = \frac{1}{R}(v_L - v_{C2})$$

Self study: substitute and rearrange the values for state variables and obtain it in matrix form

Converting a Transfer Function to State Space

Phase variables: A set of state variables where each state variable is defined to be the derivative of the previous state variable.

Consider a differential equation, $\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = b_0 u$

Choose the output, $y(t)$, and its derivatives as the state variables, x_i.

$$x_1 = y \qquad x_2 = \frac{dy}{dt} \qquad x_3 = \frac{d^2 y}{dt^2} \qquad . \qquad x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

Differentiating ⟹

$$\dot{x}_1 = \frac{dy}{dt} \qquad \dot{x}_2 = \frac{d^2 y}{dt^2} \qquad \dot{x}_3 = \frac{d^3 y}{dt^3} \qquad . \qquad \dot{x}_n = \frac{d^n y}{dt^n}$$

Substituting ⟹

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$.$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_0 x_1 - a_1 x_2 \ldots - a_{n-1} x_n + b_0 u$$

What is output equation?

Converting a Transfer Function to State Space$_{-1}$

Example: with constant term in numerator

Problem: Find the state-space representation in phase-variable form for the transfer function:

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Solution:

Step 1: Find the associated differential equation.

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$
$$\Rightarrow \dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r \quad \text{(AZIC)}$$

AZIC: Assuming Zero Initial Conditions

Step 2: Select the state variables.

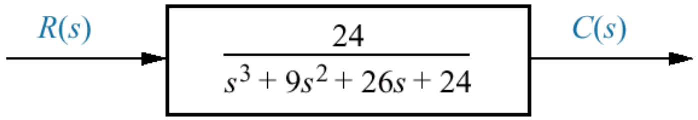
$$x_1 = c \qquad x_2 = \dot{c} \qquad x_3 = \ddot{c}$$

Differentiating $\Rightarrow$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$
$$y = c = x_1$$

Converting a Transfer Function to State Space$_{-1}$'

Example: with constant term in numerator



(a)

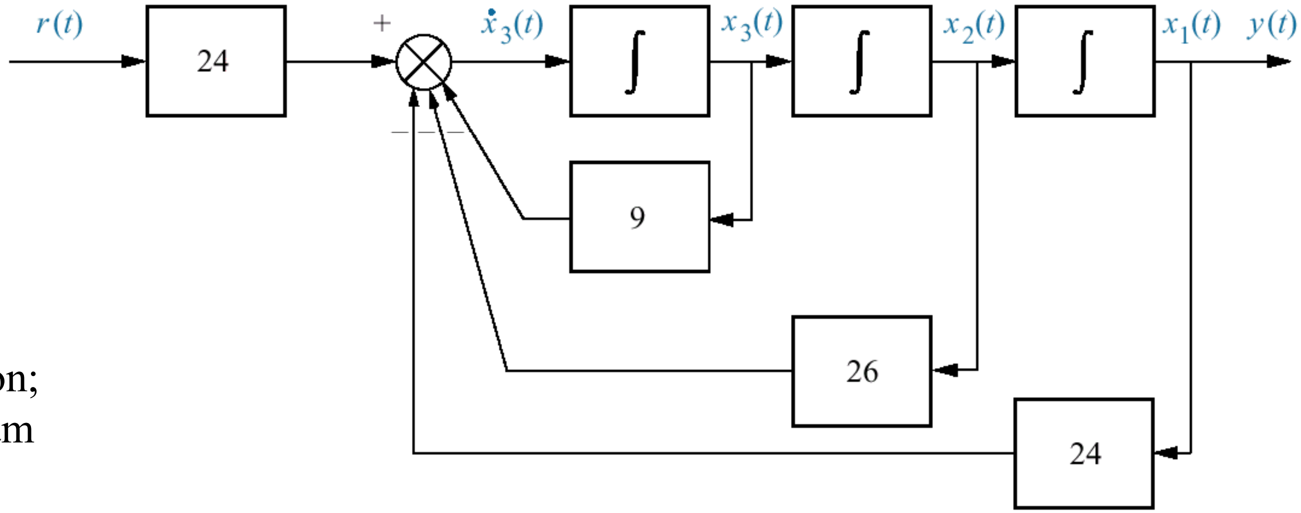


Figure: a. Transfer function;
b. equivalent block diagram
showing phase variables
(from previous slide)

(b)

Converting a Transfer Function to State Space$_{-2}$

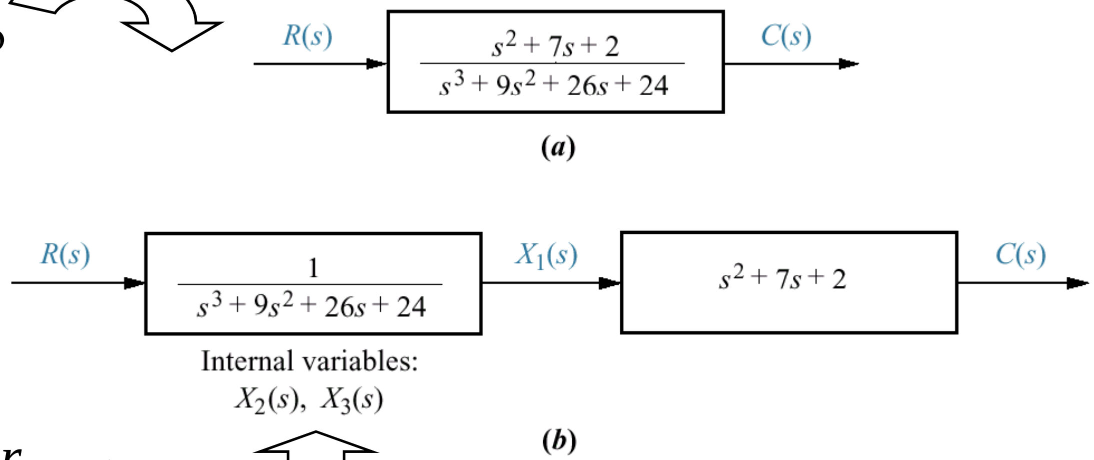
Example: with polynomial in numerator

Problem: Find the state-space representation in phase-variable form for the transfer function:

$$\frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution:

Step 1: Separate the system into two cascaded blocks as shown in figure.



Step 2: State equation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

1 / 24 of the previous example.

Converting a Transfer Function to State Space$_{-2}$'

Step 3: Introduce the effect of the block with the numerator.

$$C(s) = (s^2 + 7s + 2)X_1(s)$$

Taking inverse Laplace transform with AZIC,

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$$

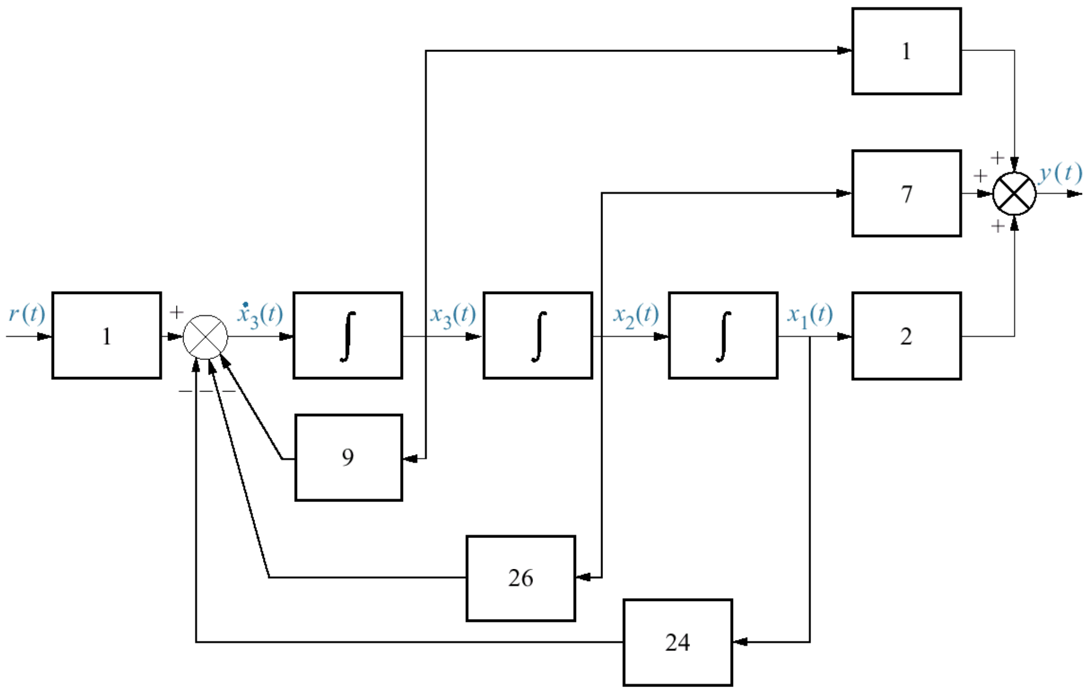
But,

$$x_1 = x_1, \quad \dot{x}_1 = x_2, \quad \ddot{x}_1 = x_3$$

Hence, $$y = c(t) = x_3 + 7x_2 + 2x_1$$

Output equation matrix form:

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



(c)

State-space Representation to Transfer Function

Problem: Given the system defined by the following state-space representation, find the transfer function, T(s) = Y(s) / U(s), where U(s) is the input and Y(s) is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$y = \mathbf{Cx} + \mathbf{Du}$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Solution: First find ($s\mathbf{I} - \mathbf{A}$):

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

State-space Representation to Transfer Function

(Continued)

Now form $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\text{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$T(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (s\mathbf{I}-\mathbf{A})^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} + \mathbf{0} = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

Homework

Chapter 3: Control Systems Engineering- by N. S. Nise

Problems:

1, 2, 3, 9, 11, 13, 14, 15