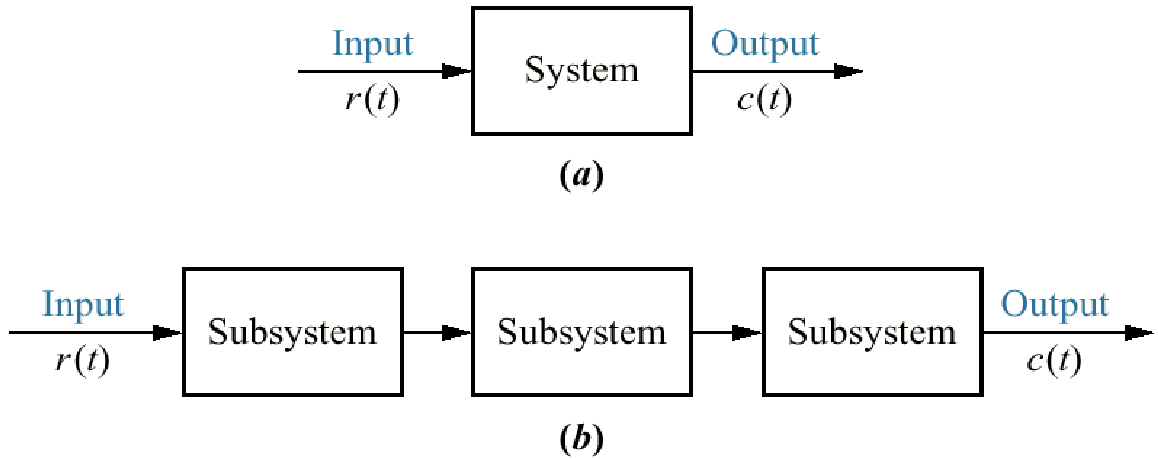
Mathematical Models of Systems

Two Methods:

(i) Transfer functions in the frequency domain,

(ii) State equations in the time domain.

a. Block diagram representation of a system;
b. block diagram representation of an interconnection of subsystems



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

Transfer function is inside each block.

Laplace Transform Review

A differential equation can describe the relationship between the input and output of a system.

Laplace transform can represent the input, output and system as separate entities.

Laplace transform can be defined as:

$$\mathcal{L}\,[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Inverse Laplace transform:

$$\mathcal{L}^{-1}\,[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds = f(t)u(t)$$

Where, $u(t) = 1 \quad t > 0$

$= 0 \quad t < 0$

Laplace Transform Table

Table 2.1
Laplace transform table

Item no.	*f(t)*	*F(s)*
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Problem: Find the Laplace transform of

$$f(t) = Ae^{-at}u(t)$$

Solution:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty Ae^{-at}e^{-st}dt$$

$$= A\int_0^\infty e^{-(s+a)t}dt = -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty}$$

$$= \frac{A}{s+a}$$

Laplace Transform Theorems

Table 2.2

Item no.	Theorem		Name
1.	$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)]$	$= F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s\to \infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

Inverse Laplace Transform

Problem: Find inverse Laplace Transform of

$$F_1(s) = \frac{1}{(s+3)^2}$$

Solution:

From item 3 and 5 of Table 2.1,

$$f_1(t) = e^{-3t}tu(t)$$

Inverse Laplace: Partial-Fraction Expansion

Case 1.

Problem:

$$F_1(s) = \frac{s^3 + 4s^2 + 5s + 4}{s^2 + 3s + 2}$$

Solution:

$$F_1(s) = s + 1 + \frac{2}{s^2 + 3s + 2}$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathfrak{I}^{-1}\left[\frac{2}{(s+1)(s+2)}\right]$$

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

To find K_1, multiply by $(s+1)$. Thus

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)}$$

Letting $s = -1$, $K_1 = 2$

Similarly, K_2 = -2

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Final solution:

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + (2e^{-t} - 2e^{-2t})u(t)$$

Laplace Transform Solution of a Differential Equation

Problem: Solve for $y(t)$, if all initial conditions are zero. $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+32y=32u(t)$

Solution: The Laplace transform is,

$$s^2Y(s)+12sY(s)+32Y(s)=\frac{32}{s}$$

$$Y(s)=\frac{32}{s(s^2+12s+32)}=\frac{32}{s(s+4)(s+8)}$$

$$=\frac{K_1}{s}+\frac{K_2}{(s+4)}+\frac{K_3}{(s+8)}$$

$$K_1=\left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0}=1$$

$$K_2=\left.\frac{32}{s(s+8)}\right|_{s\to -4}=-2$$

$$K_3=\left.\frac{32}{s(s+4)}\right|_{s\to -8}=1$$

Hence, $Y(s)=\frac{1}{s}-\frac{2}{(s+4)}+\frac{1}{(s+8)}$

Taking inverse Laplace transform, we get

$$y(t)=(1-2e^{-4t}+e^{-8t})u(t)$$

Inverse Laplace: Partial-Fraction Expansion

Case 2.

Problem: Find inverse Laplace transform of $$F(s) = \frac{2}{(s+1)(s+2)^2}$$

Solution:

$$F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

$$K_1 = \left.\frac{2}{(s+2)^2}\right|_{s \to -1} = 2$$

To find K_2, multiply by $(s+2)^2$

$$\frac{2}{(s+1)} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \quad (1)$$

Letting $s \to -2$, $K_2 = -2$

To find K_3, differentiate (1) w.r.t. s:

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3$$

Letting $s \to -2$, $K_3 = -2$

Therefore, inverse Laplace transform is:

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Inverse Laplace: Partial-Fraction Expansion

Case 3.

Problem: Find inverse Laplace transform of $F(s)=\dfrac{3}{s(s^2+2s+5)}$

Solution:

$$\frac{3}{s(s^2+2s+5)}=\frac{K_1}{s}+\frac{K_2 s+K_3}{s^2+2s+5} \quad (1)$$

$$\left.\frac{3}{s^2+2s+5}\right|_{s\to 0}=\frac{3}{5}\equiv K_1$$

Multiplying (1) by $s(s^2+2s+5)$, and putting $K_1 = 3/5$

$$3=\left(K_2+\frac{3}{5}\right)s^2+\left(K_3+\frac{6}{5}\right)s+3$$

Balancing coefficients $K_2=-\dfrac{3}{5}$, and $K_3=-\dfrac{6}{5}$

Hence, $F(s)=\dfrac{3/5}{s}-\dfrac{3}{5}\dfrac{s+2}{s^2+2s+5}$

$$\Im[Ae^{-at}\cos\omega t]=\frac{A(s+a)}{(s+a)^2+\omega^2}$$

$$\Im[Be^{-at}\sin\,\omega t]=\frac{B\omega}{(s+a)^2+\omega^2}$$

Adding,

$$\Im[Ae^{-at}\cos\omega t+Be^{-at}\sin\,\omega t]=\frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

$$F(s)=\frac{3/5}{s}-\frac{3}{5}\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2}$$

$$f(t)=\frac{3}{5}-\frac{3}{5}e^{-t}(\cos 2t+\frac{1}{2}\sin\,2t)$$

Transfer Function

General nth order, linear time-invariant differential equation:

c: output, r: input

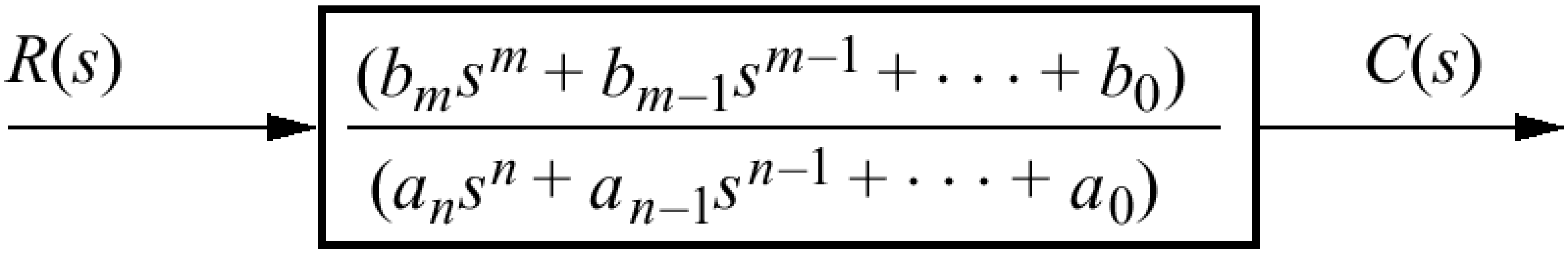
$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \ldots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \ldots + b_0 r(t)$$

Taking Laplace transform,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \ldots + a_0 C(s) + [\text{Initial condition is zero}]$$
$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \ldots + b_0 R(s) + [\text{Initial condition is zero}]$$

Transfer function:

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0)}$$



Block diagram of a transfer function

Transfer Function for a Differential Equation

Problem 1: Find the transfer function represented by $$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Solution:

Taking Laplace transform and assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s)$$

Transfer function, G(s), $$\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Problem 2: Find the transfer function represented by $$\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$$

Solution:

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$

$$\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

Problem Solving

Problem: Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

Solution:

$$C(s) = R(s)G(s) = \frac{1}{s^2} \times \frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)}$$

$$\Rightarrow C(s) = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

$$A = \left.\frac{1}{(s+4)(s+8)}\right|_{s\to 0} = \frac{1}{32}$$

$$B = \left.\frac{1}{s(s+8)}\right|_{s\to -4} = -\frac{1}{16}$$

$$C = \left.\frac{1}{s(s+4)}\right|_{s\to -8} = \frac{1}{32}$$

Hence,

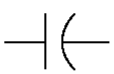
$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

Electric Network Transfer Functions

Apply the transfer function to the mathematical modeling of electronic circuits including passive networks and OAmp circuits.

Table 2.3

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

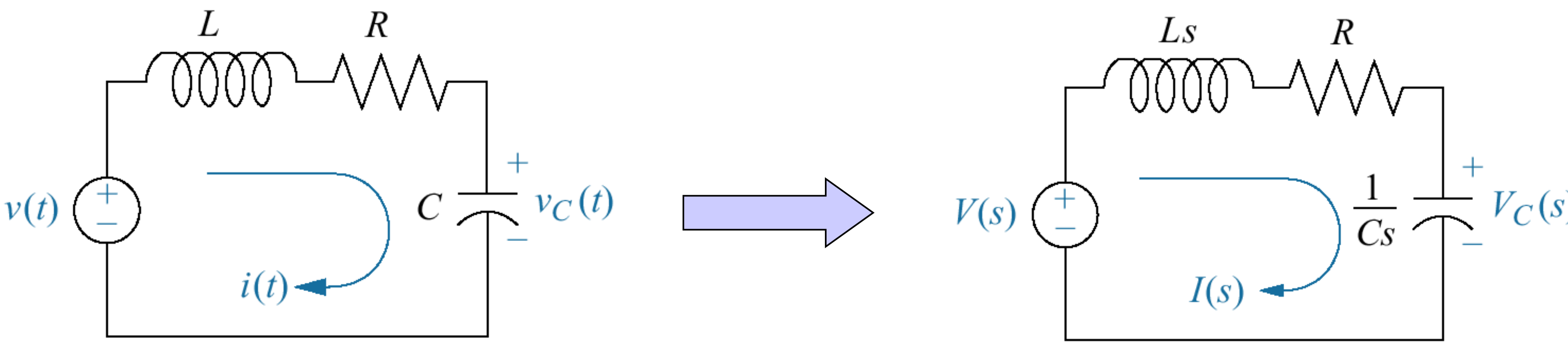
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Transfer Function: Single Loop

1

Problem: Find the transfer function relating capacitor voltage, $V_C(s)$, to input voltage, $V(s)$.



RLC network

Laplace-transformed network

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$$

$$\Rightarrow \frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

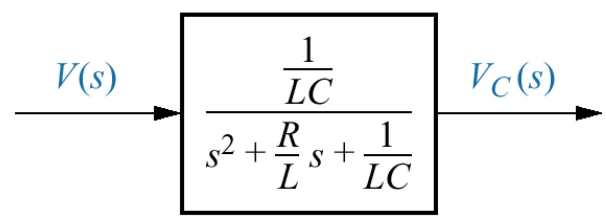
We know, $V_C(s) = I(s)\frac{1}{Cs}$

$$\Rightarrow V_C(s) = \frac{V(s)}{Ls + R + \frac{1}{Cs}} \times \frac{1}{Cs}$$

$$\Rightarrow \frac{V_C(s)}{V(s)} = \frac{Cs}{s^2 LC + sRC + 1} \times \frac{1}{Cs}$$

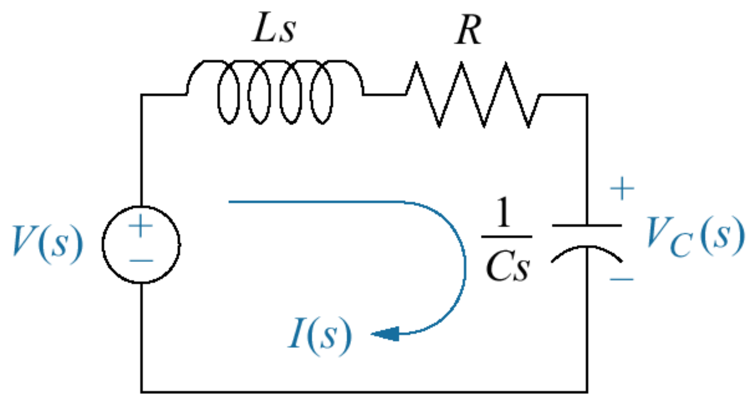
$$= \frac{1}{s^2 LC + sRC + 1}$$

$$= \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$



Transfer Function: Single Node

2



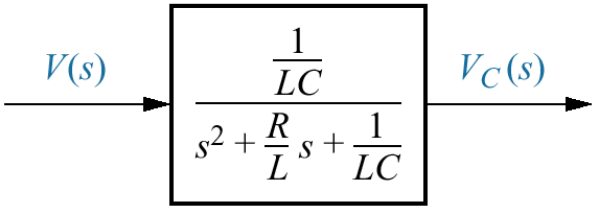
$$I_C(s) + I_{RL}(s) = 0$$

$$\Rightarrow \frac{V_C(s)}{\frac{1}{Cs}} + \frac{V_C(s) - V(s)}{R + Ls} = 0$$

$$\Rightarrow V_C(s)\left(Cs + \frac{1}{R + Ls}\right) = \frac{V(s)}{R + Ls}$$

$$\Rightarrow \frac{V_C(s)}{V(s)} = \frac{1}{s^2 LC + sRC + 1}$$

$$= \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

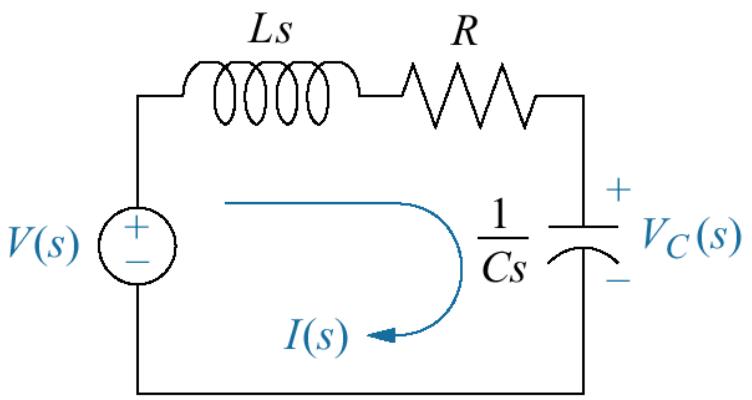


Transfer Function: Single Loop via Voltage Division

3

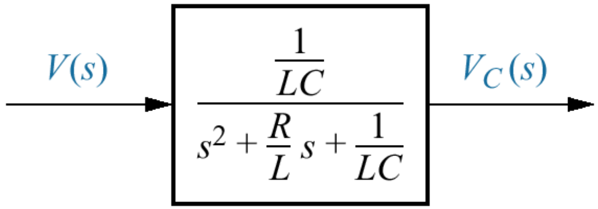
Voltage across capacitor is some proportion of the input voltage.

$$\frac{\text{Impedance of the capacitor}}{\text{Sum of the impedances}}$$



$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

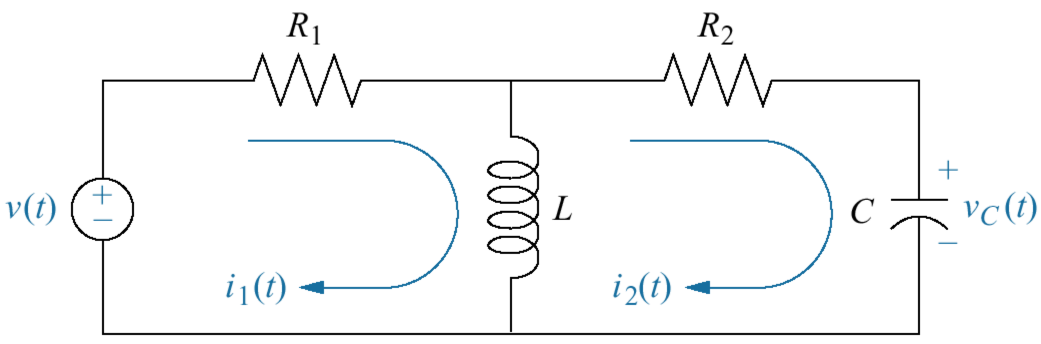
$$\Rightarrow \frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$



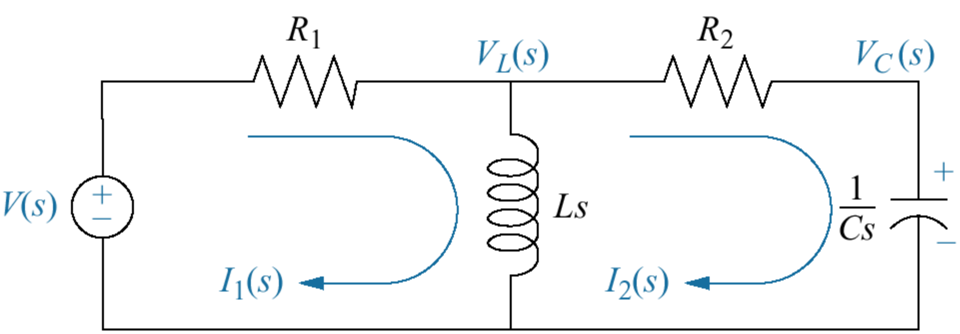
$$\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Which one is the easiest? 1, 2, or 3?

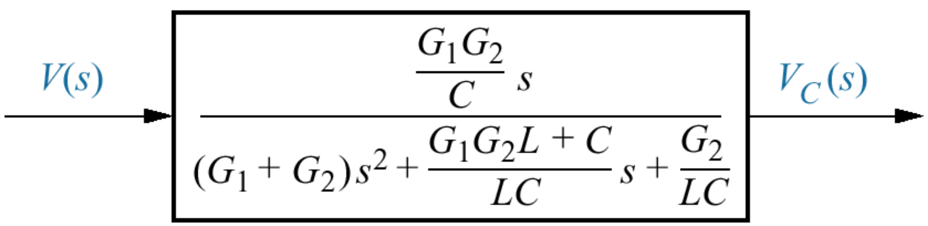
Complex Circuits via Nodal Analysis



(a)



(b)



Current from $V_L(s)$

$$\frac{V_L(s)-V(s)}{R_1}+\frac{V_L(s)}{Ls}+\frac{V_L(s)-V_C(s)}{R_2}=0$$

$$CsV_C(s)+\frac{V_C(s)-V_L(s)}{R_2}=0$$

Current from $V_C(s)$

Expressing resistances as conductances, G=1/R

$$\left(G_1+G_2+\frac{1}{Ls}\right)V_L(s)-G_2V_C(s)=V(s)G_1$$

$$-G_2V_L(s)+(G_2+Cs)V_C(s)=0$$

Transfer function:

$$\frac{V_C(s)}{V(s)}=\frac{\frac{G_1G_2}{C}s}{(G_1+G_2)s^2+\frac{G_1G_2L+C}{LC}s+\frac{G_2}{LC}}$$

Mesh Equations via Inspection

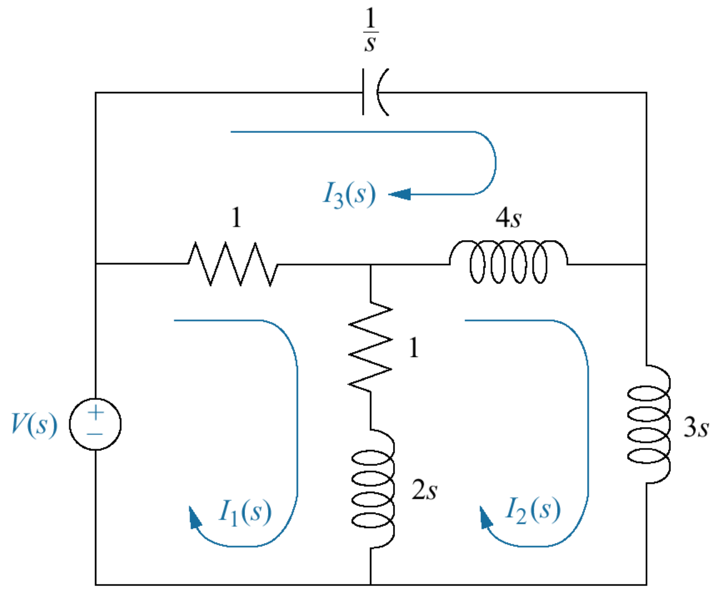
For Mesh 1:

(Sum of impedances around Mesh 1)×$I_1(s)$

- (Sum of impedances common to Mesh 1 and Mesh 2)×$I_2(s)$

- (Sum of impedances common to Mesh 1 and Mesh 3)×$I_3(s)$

= (Sum of applied voltages around Mesh 1)

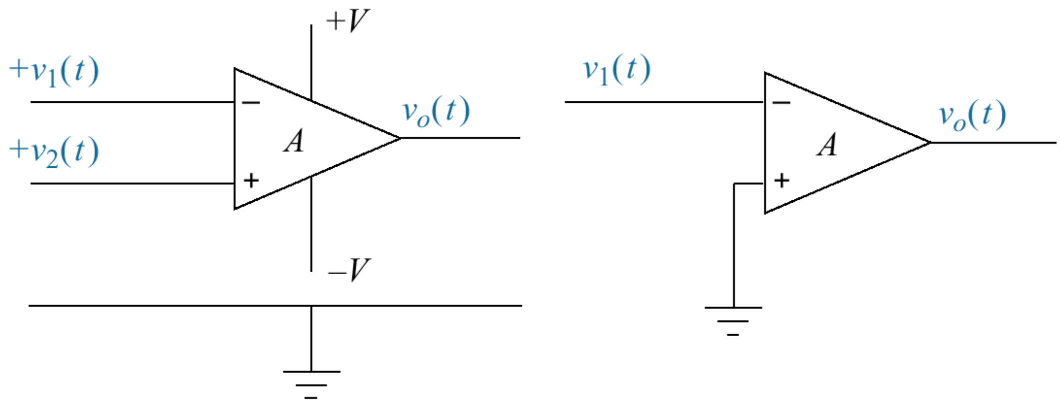


Three-loop electrical network

$$+(2s+2)I_1(s)-(2s+1)I_2(s)-I_3(s)=V(s)$$
$$-(2s+1)I_1(s)+(9s+1)I_2(s)-4sI_3(s)=0$$
$$-I_1(s)-4sI_2(s)+\left(4s+1+\frac{1}{s}\right)I_3(s)=0$$

Operational Amplifier

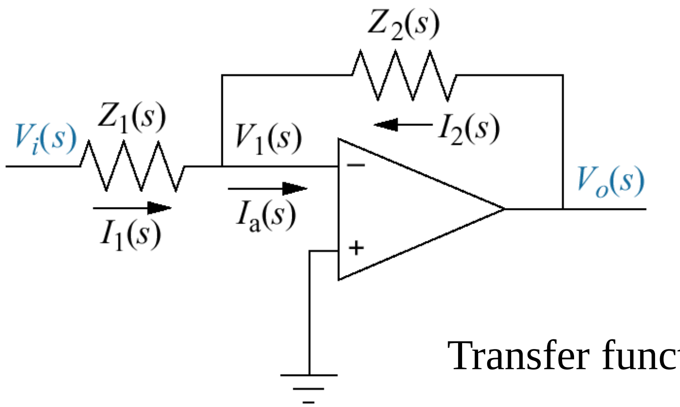
a. Operational amplifier;
b. schematic for an inverting operational amplifier;



(a) **(b)**

$$v_0(t) = A(v_2(t) - v_1(t))$$

c. inverting operational amplifier configured for transfer function realization. Typically, the amplifier gain, A, is omitted.



(c)

Transfer function:

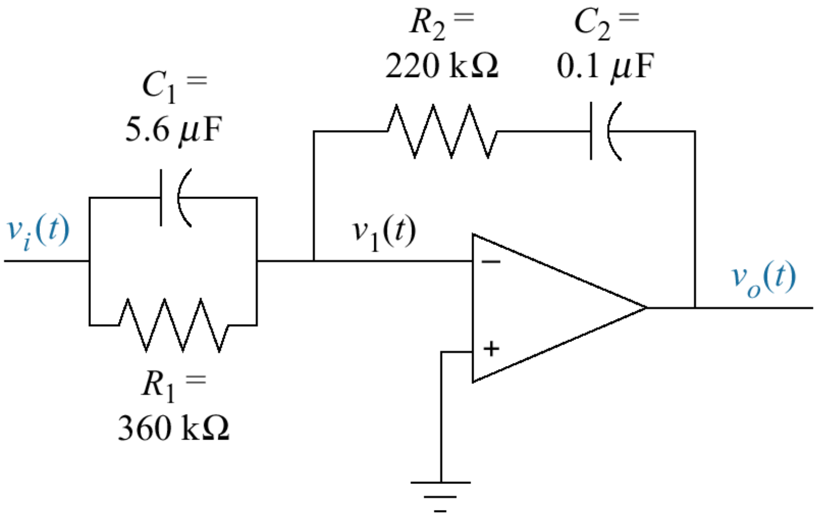
$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

$I_1(s) = -I_2(s)$, as $I_a(s) = 0$, because of high input impedance

Problem Solving
Inverting Operational Amplifier

Problem: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit below.

Solution:



For parallel components, $Z_1(s)$ is the reciprocal of the sum of the admittances.

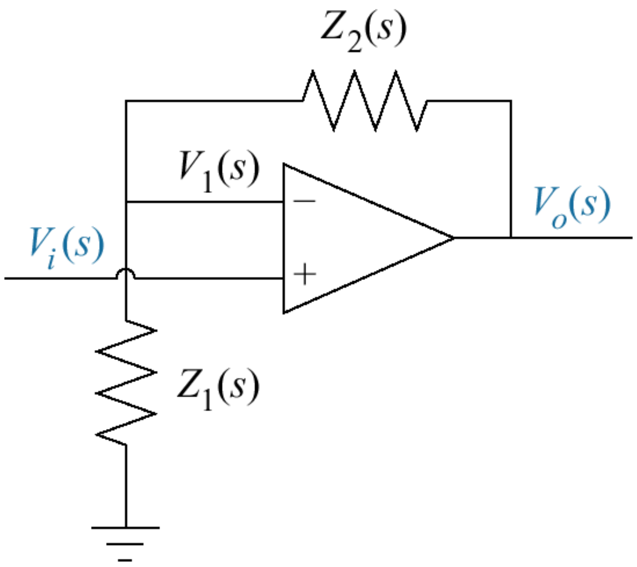
$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6\times10^{-6}s + \frac{1}{360\times10^3}}$$

For serial components, $Z_2(s)$ is the sum of the impedances.

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220\times10^3 + \frac{10^7}{s}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -1.232\times\frac{s^2 + 45.95s + 22.55}{s}$$

Non-inverting Operational Amplifier



$$V_o(s) = A(V_i(s) - V_1(s))$$

Using voltage division,

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s)$$

$$\therefore \frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))}$$

For large A, we disregard '1' in the denominator.

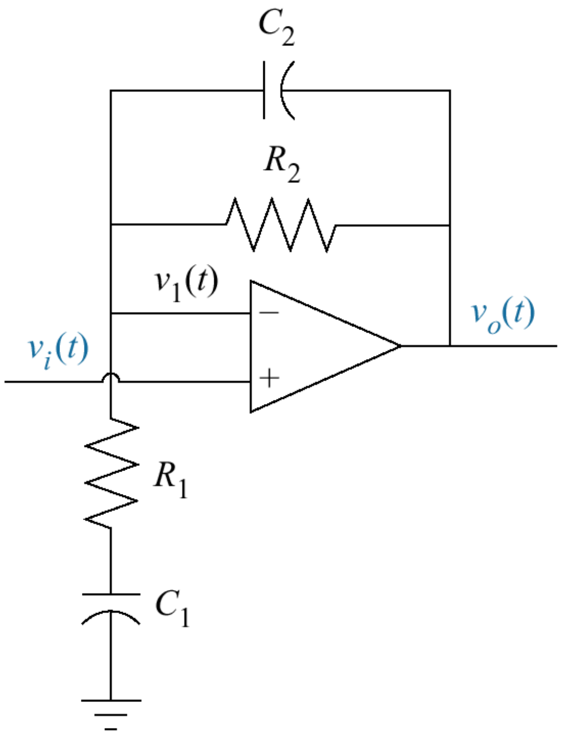
$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

Problem Solving

Non-Inverting Operational Amplifier

$$Z_1(s) = R_1 + \frac{1}{C_1 s}$$

$$Z_2(s) = \frac{1}{C_2 s + \frac{1}{R_2}} = \frac{R_2}{R_2 C_2 s + 1} = \frac{R_2 (1/C_2 s)}{R_2 + (1/C_2 s)}$$

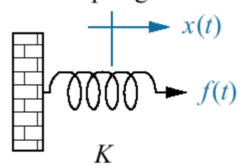
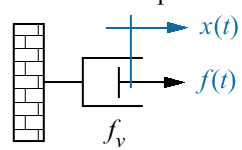
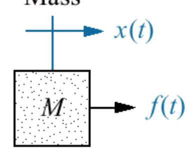


Now use the following equation:

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

Translational Mechanical System Transfer Functions

Table 2.4
Force-velocity, force-displacement, and impedance translational relationships
for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

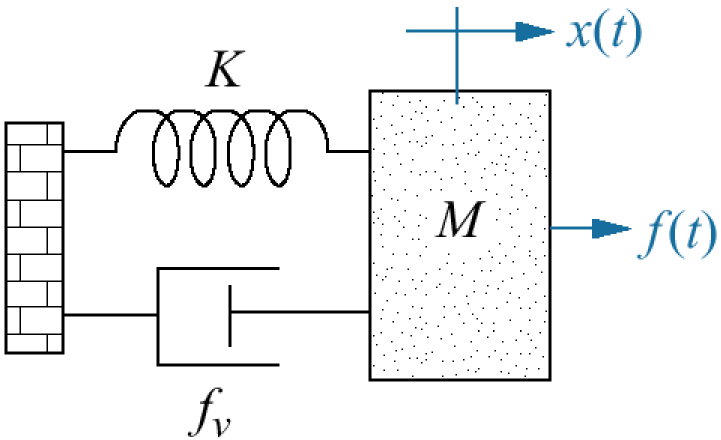
Mechanical systems have three passive, linear components: **Spring**, **Mass**, **Viscous Damper**.

K: Spring constant

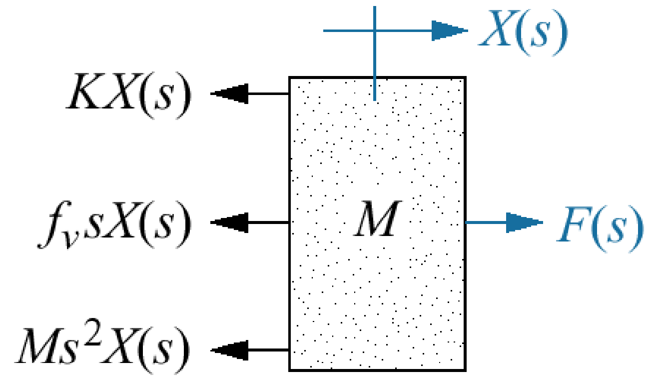
f_v: Coefficient of viscous friction

M: Coefficient of mass

Transfer Functions: One Degree of Freedom



Mass, spring, and damper system

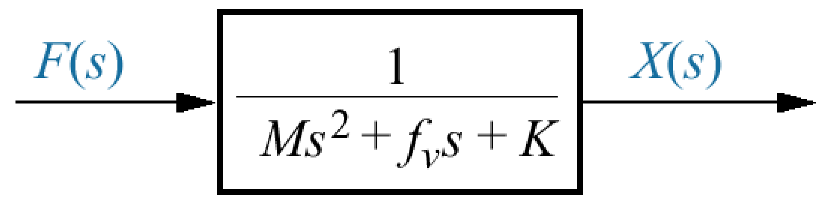


Transformed free body diagram

Sum of impedances × X(s) = Sum of applied forces

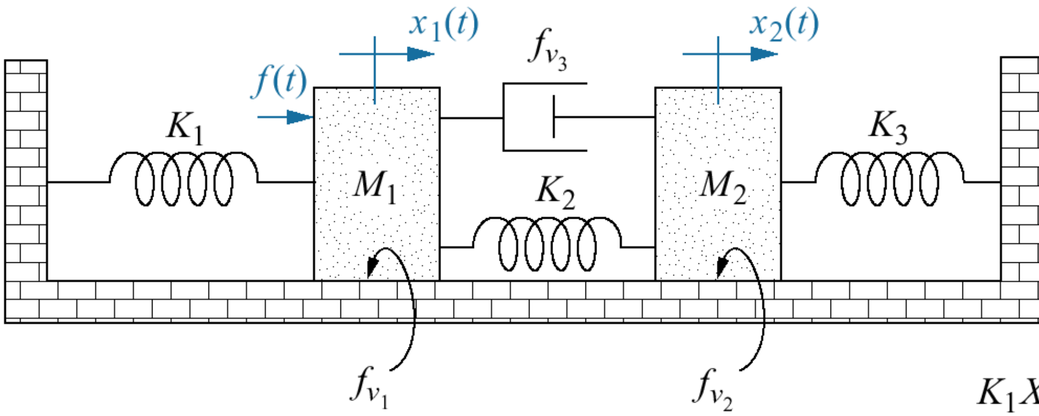
$$(K + f_v s + Ms^2) \times X(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

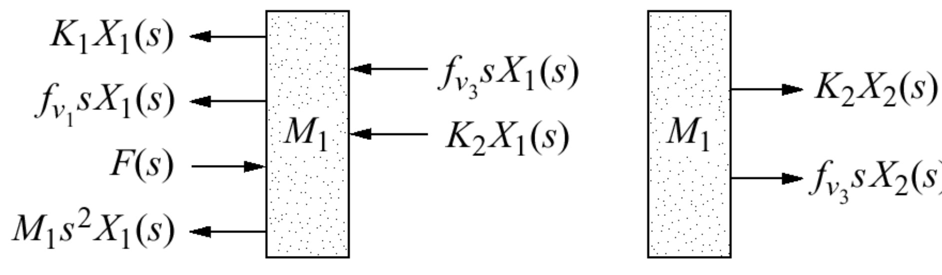


Transfer function

Transfer Functions: Two Degrees of Freedom

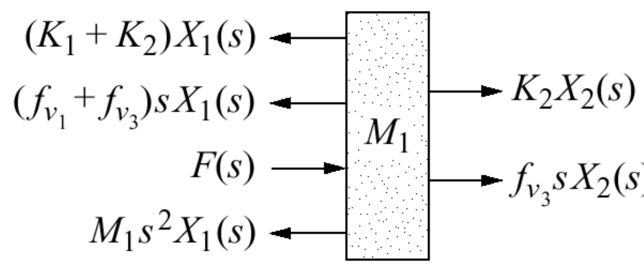


Two-degrees-of-freedom translational mechanical system



(a) (b)

a. Forces on M1 due only to motion of M1
b. forces on M1 due only to motion of M2
c. all forces on M1

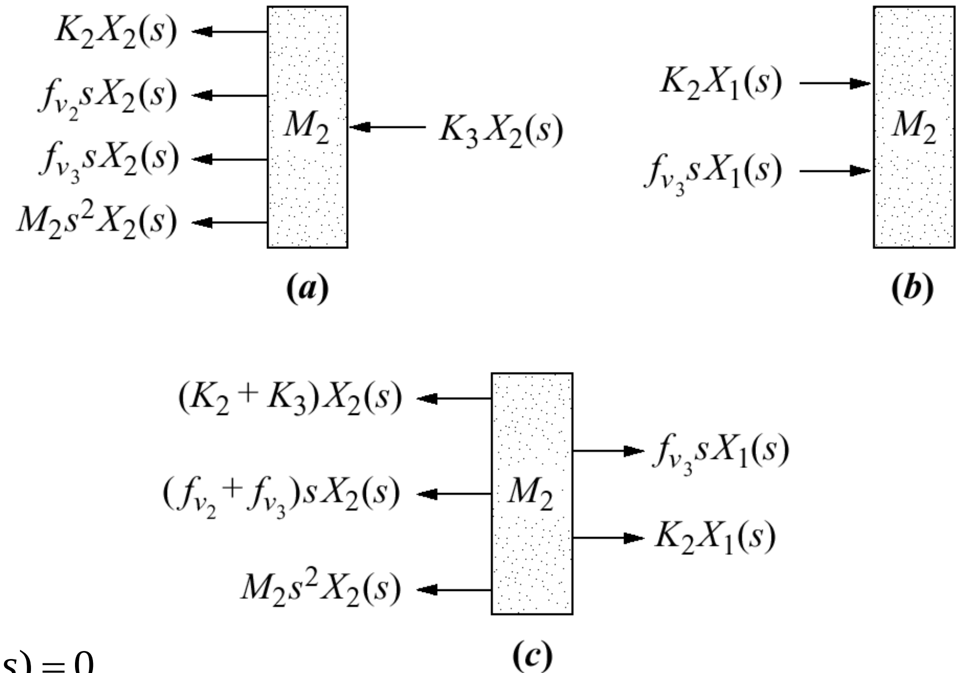


(c)

$$[M_1s^2+(f_{v1}+f_{v3})s+(K_1+K_2)]X_1(s)-(f_{v3}s+K_2)X_2(s)=F(s)$$

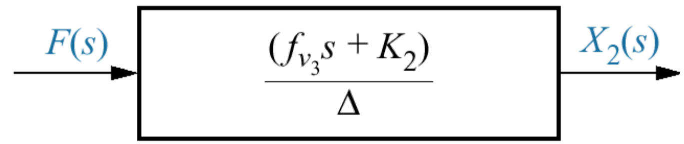
Transfer Functions: Two Degrees of Freedom Continued

a. Forces on M_2 due only to motion of M_2;
b. forces on M_2 due only to motion of M_1;
c. all forces on M_2

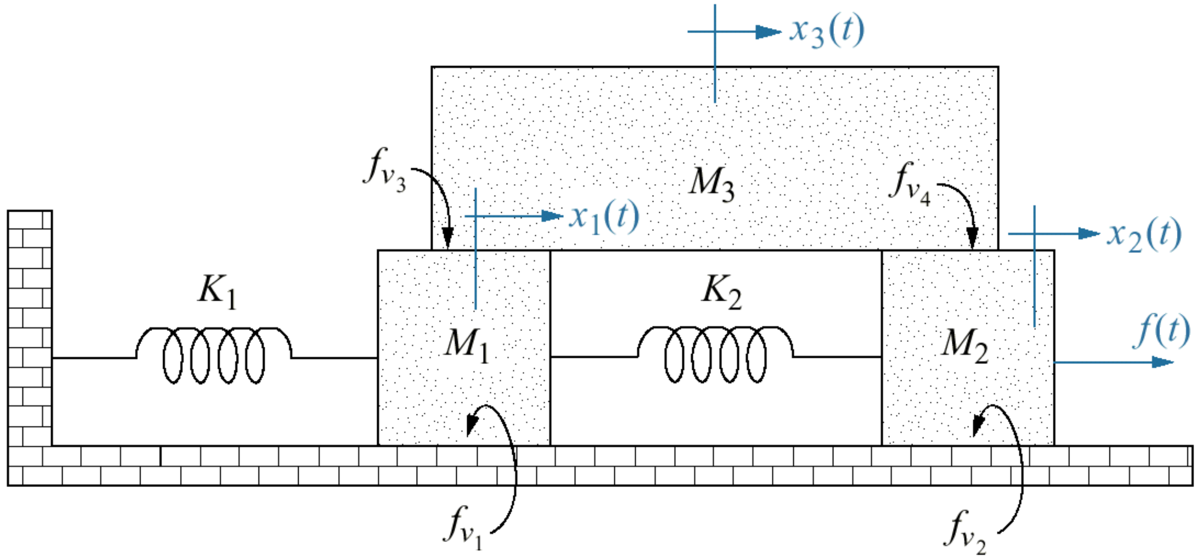


$$-(f_{v3}s+K_2)X_1(s)+[M_2s^2+(f_{v2}+f_{v3})s+(K_2+K_3)]X_2(s)=0$$

Transfer function: $\dfrac{X_2(s)}{F(s)}$



Transfer Functions: Three Degrees of Freedom



M

M_1: $$[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v3} s X_3(s) = 0$$

M_2: $$-K_2 X_1(s) + [M_2 s^2 + (f_{v2} + f_{v4})s + K_2]X_2(s) - f_{v4} s X_3(s) = F(s)$$

M_3: $$-f_{v3} s X_1(s) - f_{v4} s X_2(s) + [M_3 s^2 + (f_{v3} + f_{v4})s]X_3(s) = 0$$

Nonlinearity

Linear systems have two properties: (1) additivity, and (2) homogeneity.

Additivity (superposition):

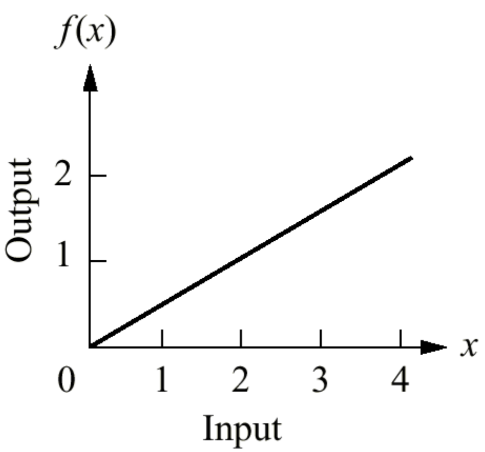
If $r_1(t) \rightarrow c_1(t)$ and $r_2(t) \rightarrow c_2(t)$, then $r_1(t) + r_2(t) \rightarrow c_1(t) + c_2(t)$

$r(t)$: Input

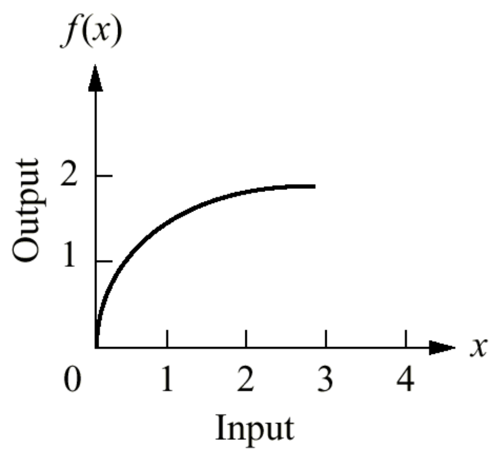
$c(t)$: Output

Homogeneity:

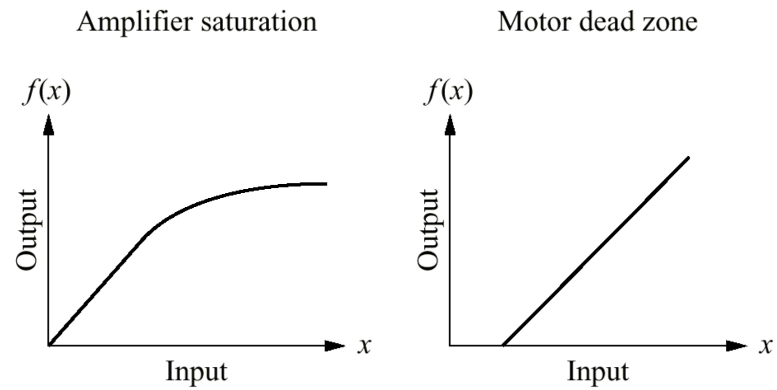
If $r_1(t) \rightarrow c_1(t)$, then $Ar_1(t) \rightarrow Ac_1(t)$



Linear system



Nonlinear system



Some physical nonlinearities

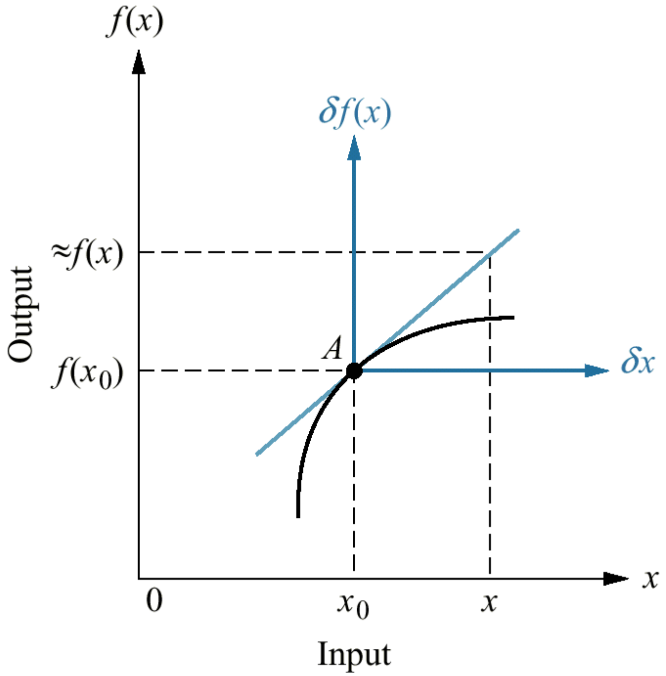
Linearization

Making linear approximation to a nonlinear system.

Linear approximation simplifies the analysis and design of a system
→ to obtain transfer functions.

Linear approximation at A = slope (m) of the curve at A

$$[f(x) - f(x_0)] \approx m_a (x - x_0)$$
$$\Rightarrow \delta f(x) \approx m_a \delta x$$
$$\Rightarrow f(x) \approx f(x_0) + m_a \delta x$$



Linearization about a point A

Linearizing a Function

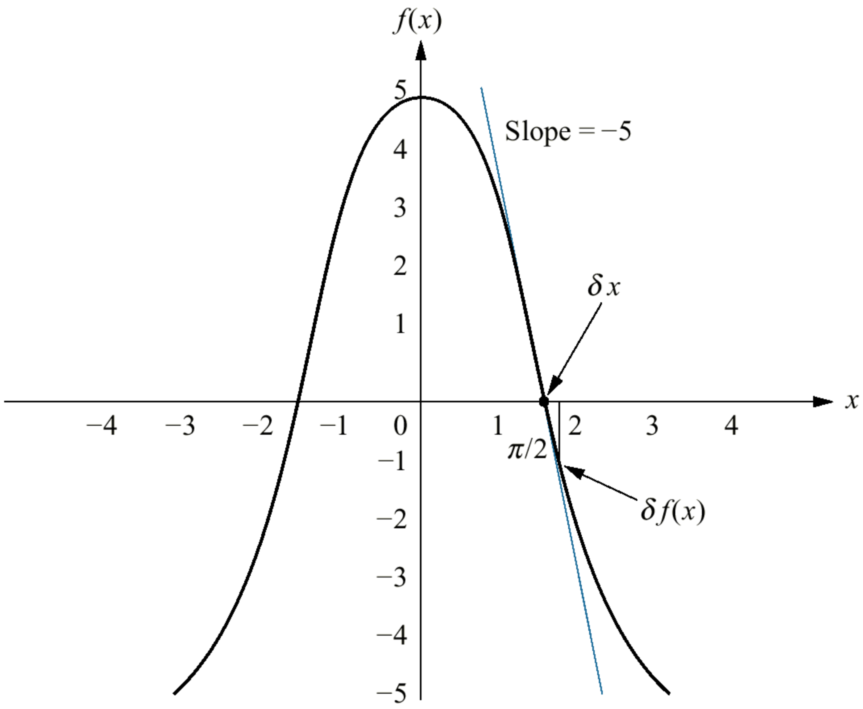
Problem: Linearize $f(\mathrm{x}) = 5\mathit{cos}\mathrm{x}$ about x = π/2.

Solution:

$$\left.\frac{df}{dx}\right|_{x=\pi/2} = -5\sin x\Big|_{x=\pi/2} = -5$$

$$f(x_0) = f(\pi/2) = 5\cos(\pi/2) = 0$$

$$f(x) = -5\delta x \text{ for small excursions of } x \text{ about } \pi/2$$



Linearizing a Differential Equation

Problem: Linearize the following equation for small excursions about $x = \pi/4$.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0$$

Nonlinear

Solution:

$$\text{Let } x = \delta x + \pi/4$$

$$\frac{d^2\left(\delta x + \frac{\pi}{4}\right)}{dt^2} + 2\frac{d\left(\delta x + \frac{\pi}{4}\right)}{dt} + \cos\left(\delta x + \frac{\pi}{4}\right) = 0$$

But, $\dfrac{d^2\left(\delta x + \frac{\pi}{4}\right)}{dt^2} = \dfrac{d^2\delta x}{dt^2}$ and $\dfrac{d\left(\delta x + \frac{\pi}{4}\right)}{dt} = \dfrac{d\delta x}{dt}$

Now, $\cos x$ can be linearized with the truncated Taylor series:

Let, $f(x) = \cos(\delta x + \pi/4)$, $f(x_0) = \cos(\pi/4)$, $(x - x_0) = \delta x$

Taylor series:

$$f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

$$\cos\left(\delta x + \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \left.\frac{d\cos x}{dx}\right|_{x=\frac{\pi}{4}} \delta x = -\sin\left(\frac{\pi}{4}\right)\delta x$$

$$\Rightarrow \cos\left(\delta x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\delta x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\delta x$$

$$\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} - \frac{1}{\sqrt{2}}\delta x = -\frac{1}{\sqrt{2}}$$

Solving for δx, $x = \delta x + \dfrac{\pi}{4}$

Homework II

Text: Control Systems Engineering – by NISE, Chapter 2.

Problems:

5, 6, 8, 10, 16, 17, 20 (a), 21, 22, 24, 27, 50.

There are many other related problems. Drill these problems also to prepare yourself for exams.