Digital System Stability via s-Plane

Bilinear transform (s-plane to z-plane and vice versa):

$$s = \frac{z+1}{z-1} \quad \text{and} \quad z = \frac{s+1}{s-1}$$

Letting, $s = \sigma + j\omega$

$$z = \frac{(\sigma+1)+j\omega}{(\sigma-1)+j\omega} \Rightarrow |z| = \frac{\sqrt{(\sigma+1)^2+\omega^2}}{\sqrt{(\sigma-1)^2+\omega^2}}$$

Thus,

|z| < 1, when σ < 0
|z| > 1, when σ > 0
|z| = 1, when σ = 0

Example:

Given, *T*(z) = *N*(z) / *D*(z)

Where, $D(z) = z^3 - z^2 - 0.2z + 0.1$

Use Routh-Hurwitz criterion to find the number of z-plane poles of T(z) inside, outside, and on the unit circle. Is the system stable?

From bilinear transform and D(z) = 0, we get:

$$s^3 - 19s^2 - 45s - 17 = 0$$

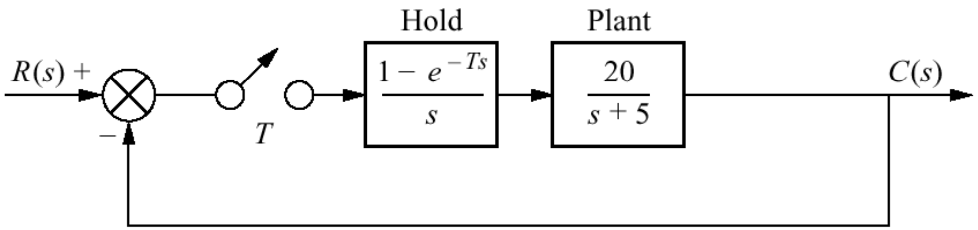
s^3	1	-45
s^2	-19	-17
s^1	-45.89	0
s^0	-17	0

From Routh table, there are one root at right half-plane and two roots on the left half-plane.

One pole outside the unit circle and two ploes inside.
The system is unstable because one pole is outside the unit circle.

Digital System Stability: Example

Problem: Determine the range of sampling interval, *T*, that will make the following system stable.



Solution:

The z- transform of the closed loop system is $T(z) = \dfrac{G(z)}{1+G(z)}$

First find G(s):

$$G(s) = 20\frac{1-e^{-Ts}}{s} \times \frac{1}{s+5} = \frac{20}{5}(1-e^{-Ts})\left(\frac{1}{s} - \frac{1}{s+5}\right)$$

$$\therefore G(z) = 4\left(\frac{z-1}{z}\right)\left(\frac{z}{z-1} - \frac{z}{z-e^{-5T}}\right) = 4\frac{1-e^{-5T}}{z-e^{-5T}}$$

$$\therefore T(z) = \frac{G(z)}{1+G(z)} = \frac{4(1-e^{-5T})}{z-5e^{-5T}+4}$$

Pole is at $(5e^{-5T}-4)$: If $T > 0.1022$ sec, the pole is outside the unit circle→ unstable system.

Therefore, for stability: 0 < *T* < 0.1022 sec

Digital System Steady-State Errors

Sampled steady-state error for unity feedback system:

$$e^*(\infty) = \lim_{z\to 1}(1 - z^{-1})\frac{R(z)}{1+G(z)}$$

Unit Step Input:

$R(s) = 1/s. \rightarrow R(z) = z/(z-1)$ ⇒ $\therefore e^*(\infty) = \dfrac{1}{1+\lim_{z\to 1} G(z)} = \dfrac{1}{1+K_p}$

K_p: Static error constant

Unit Ramp Input:

$$R(z) = \frac{Tz}{(z-1)^2}$$

⇒ $e^*(\infty) = \dfrac{1}{K_v}$, where $K_v = \dfrac{1}{T}\lim_{z\to 1}(z-1)G(z)$

K_v: Static error constant

Unit Parabolic Input:

$$R(z) = \frac{T^2 z(z+1)}{2(z-1)^3}$$

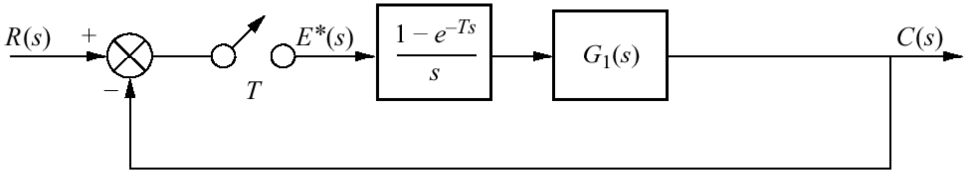
⇒ $e^*(\infty) = \dfrac{1}{K_a}$, where $K_a = \dfrac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z)$

K_a: Static error constant

When the errors will be zero? For example: If G(z) has two poles at z = 1, then error will be zero form unit ramp input.

Finding Steady-State Errors

Problem: For step, ramp, and parabolic inputs, find the steady-state error for the feedback control system with:



$$G_1(s) = \frac{10}{s(s+1)}$$

Solution:

$$G(s) = \frac{10(1-e^{-Ts})}{s^2(s+1)} = 10(1-e^{-Ts})\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$$\therefore G(z) = 10(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= 10\left[\frac{T}{z-1} - 1 + \frac{z-1}{z-e^{-T}}\right]$$

How to check stability:

Find: $T(z) = \dfrac{G(z)}{1+G(z)}$

Put value of *T* (given), and find poles. If the poles are inside the unit circle, then the system is stable.

Skill Assessment Ex: 13.7

For a step input: $K_p = \lim_{z\to 1} G(z) = \infty; \quad e^*(\infty) = \dfrac{1}{1+K_p} = 0$

For a ramp input: $K_v = \lim_{z\to 1}(z-1)G(z) = 10; \quad e^*(\infty) = \dfrac{1}{K_v} = 0.1$

For a parabolic input: $K_a = \dfrac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z) = 0; \quad e^*(\infty) = \dfrac{1}{K_a} = \infty$

Transient Response on the z-Plane

Vertical lines in s-plane: constant settling time → $s = \sigma_1 + j\omega$, where $\sigma_1 = -4/T_s$

$$z = e^{sT} \Rightarrow e^{\sigma_1 T} e^{j\omega T} = r_1 e^{j\omega T}$$

$$e^{\sigma_1 T} = e^{-4/(T_s/T)} = r \Rightarrow \frac{T_s}{T} = -\frac{4}{\ln(r)}$$

Circle in z-plane

Horizontal lines in s-plane: constant peak time.

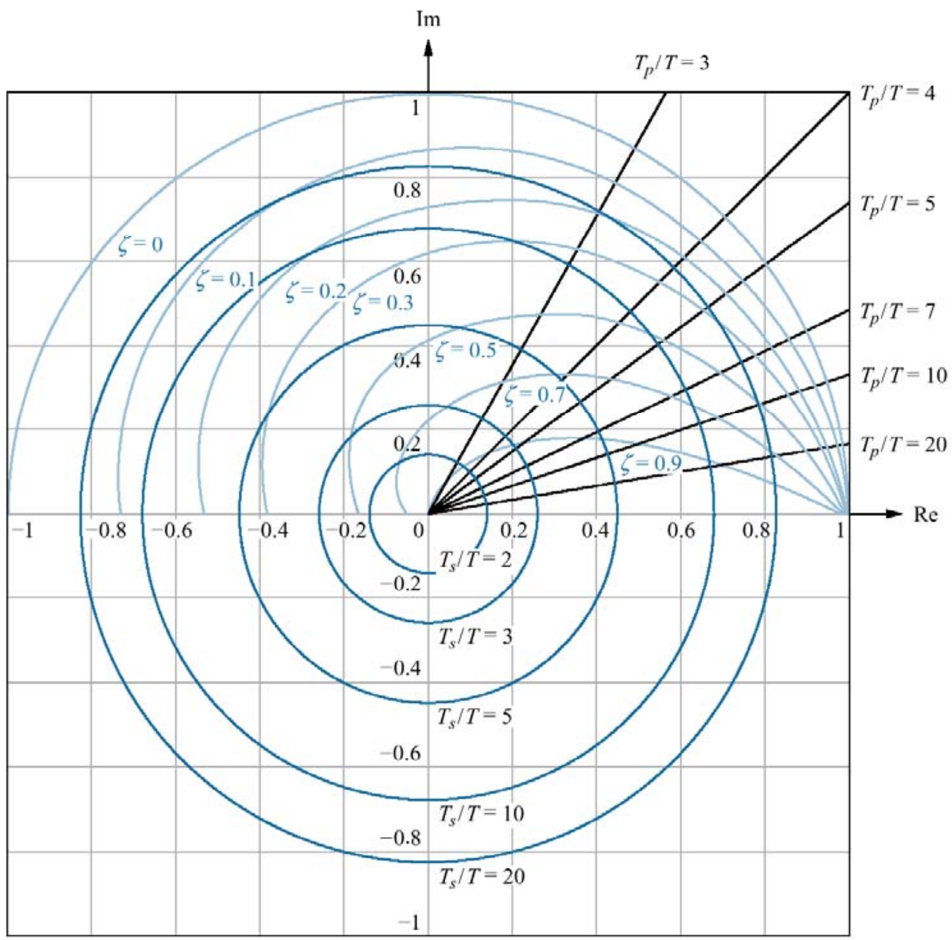
$s = \sigma + j\omega_1$, where $\omega_1 = \pi/T_p$

$$z = e^{sT} \Rightarrow e^{\sigma T} e^{j\omega_1 T} = e^{\sigma T} e^{j\theta_1}$$

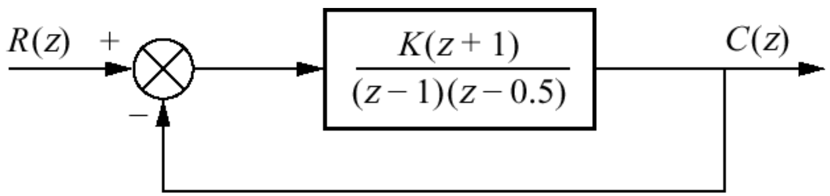
$$\omega_1 T = \theta_1 = \frac{\pi}{(T_p/T)} \Rightarrow \frac{T_p}{T} = \frac{\pi}{\theta}$$

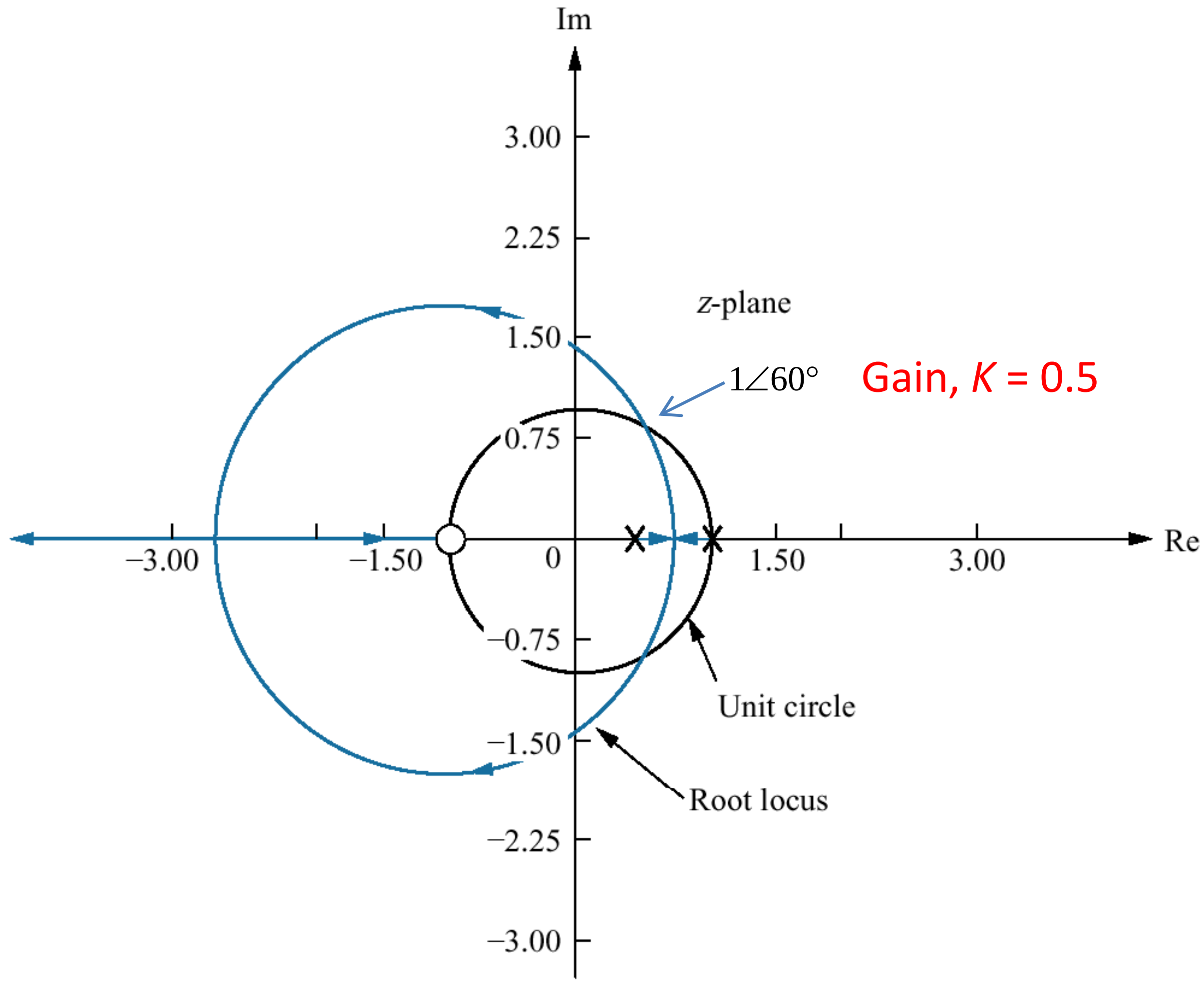
Angle of Radial line in z-plane

Also check for constant damping ratio lines.



Stability Design via Root Locus (z-plane)





Stable for $0 < K < 0.5$

Transient Response Design via Gain Adjustment

Problem: Find the value of gain, *K*, to yield a damping ratio of 0.7.

Solution:

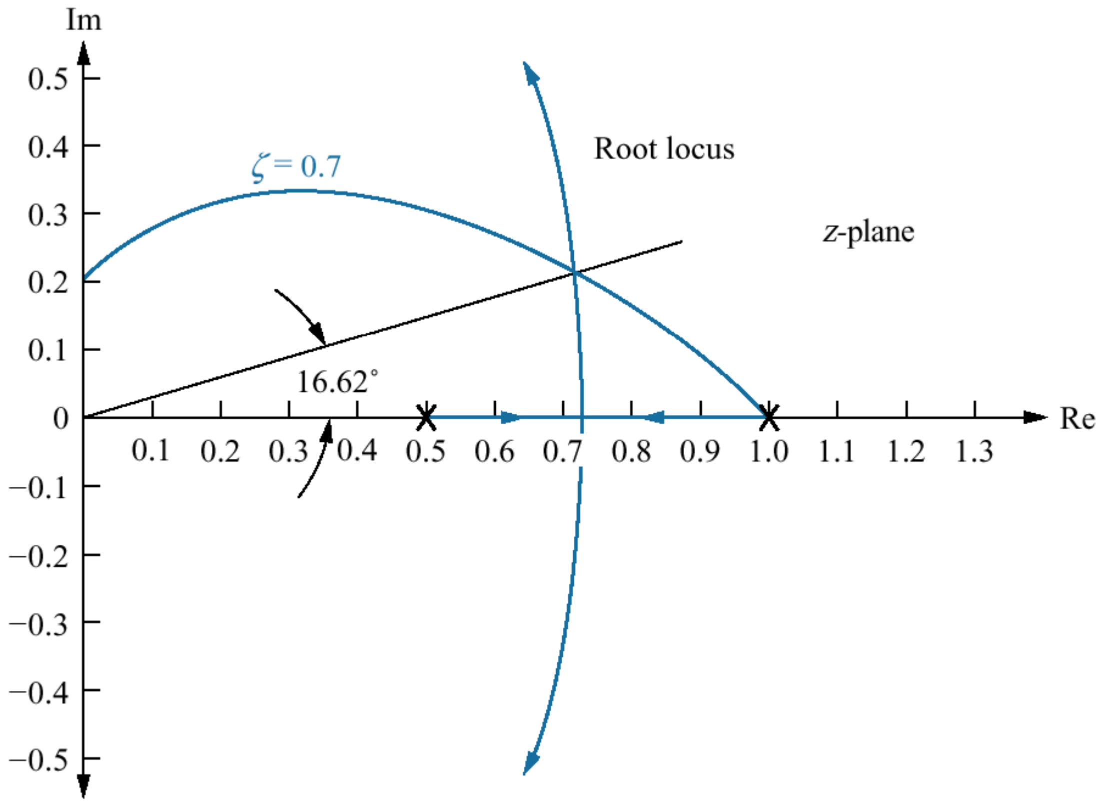
Draw damping ratio 0.7 line.

Find the intersection between the damping ratio line and root locus of the previous example.

Draw a radial line from the origin to that intersection point. (16.62° line).

Intersection line; 0.719 + *j*0.215.

Find *K*. (*K* = 0.06627)



Skill Assessment Ex: 13.8

Cascade Compensation via s-Plane

Bilinear transform (between *s* and *z*) at sampling instant: $s = \dfrac{2(z-1)}{T(z+1)}$ Tustin Transformation

$$z = \frac{-\left(s + \frac{2}{T}\right)}{\left(s - \frac{2}{T}\right)} = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

Problem: $G_p(s) = \dfrac{1}{s(s+6)(s+10)}$

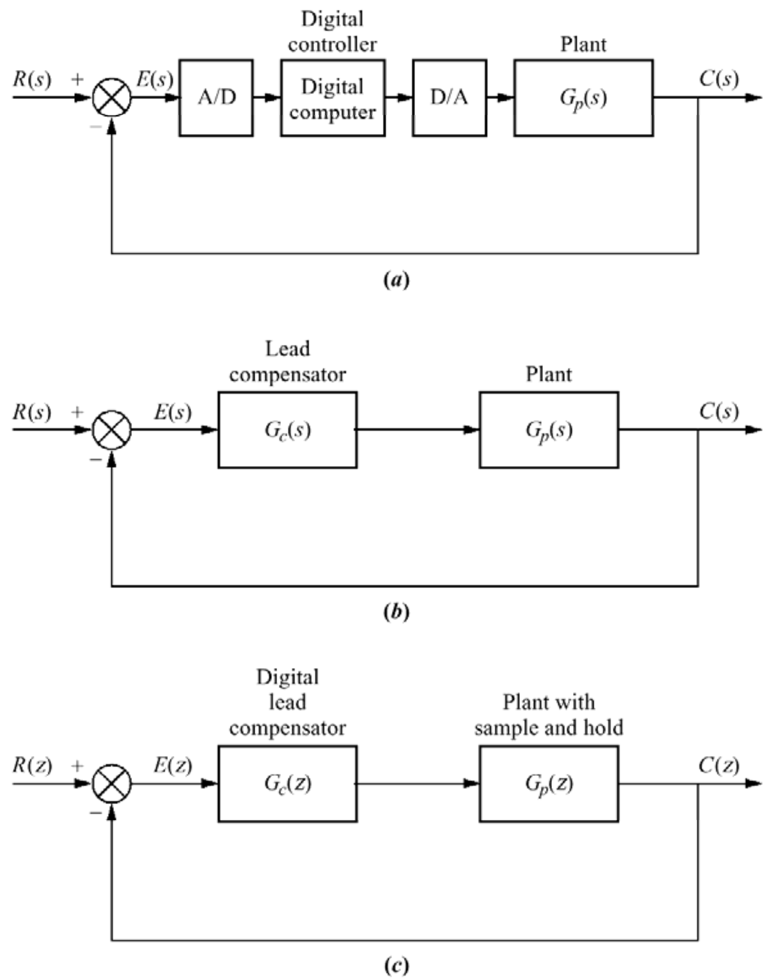
Overshoot = 20%, T_s = 1.1 sec.

Design a digital lead compensator, $G_c(z)$.

Solution: From previous example: $G_c(s) = \dfrac{1977(s+6)}{(s+29.1)}$

Max T should be $0.15/\omega < T < 0.5/\omega$

ω = zero dB frequency response. (HOW?)



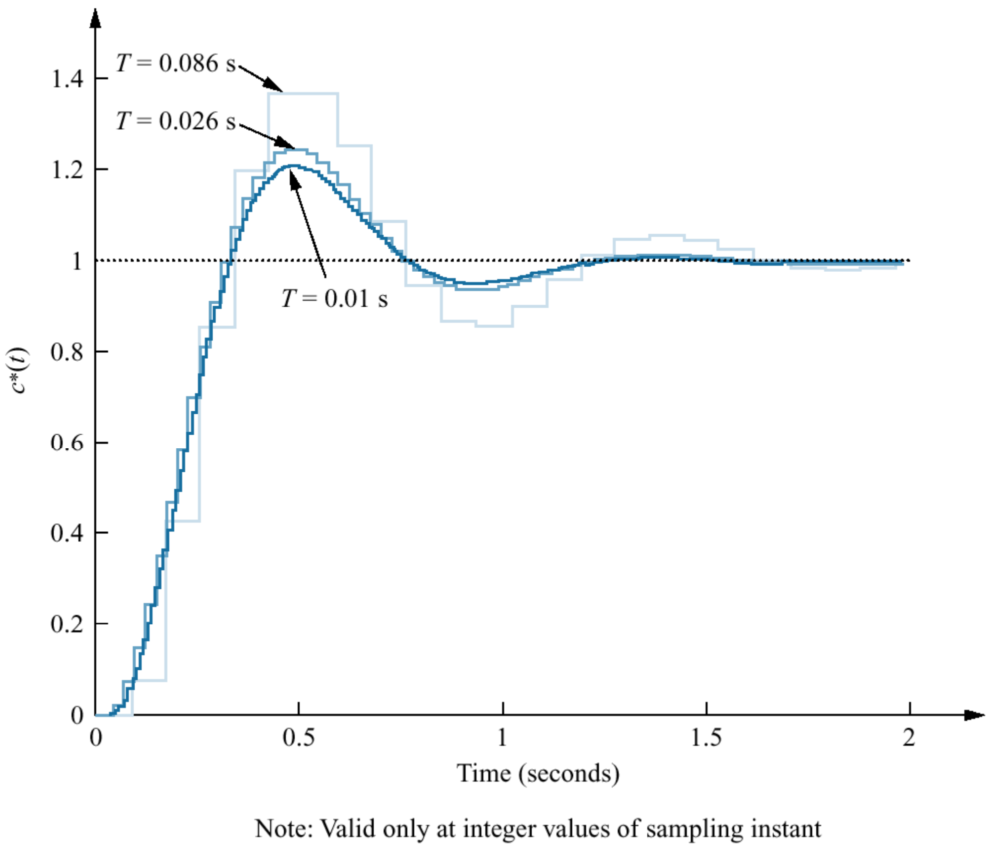
Cascade Compensation

First find: $G_p(s)G_c(s)$. Then put $s = j\omega$. Solve $|Gp(\omega)Gc(\omega)| = 0$.

We will find $\omega = 5.8$ rad/sec. Let use $T = 0.01$ sec.

After substitution, we get: $G_c(z) = \dfrac{1778z - 1674}{z - 0.746}$

Find $G_p(z)$ from $G_p(s)$ and zero-order hold:



Skill Assessment Ex: 13.9

Digital Cascade Compensator Implementation

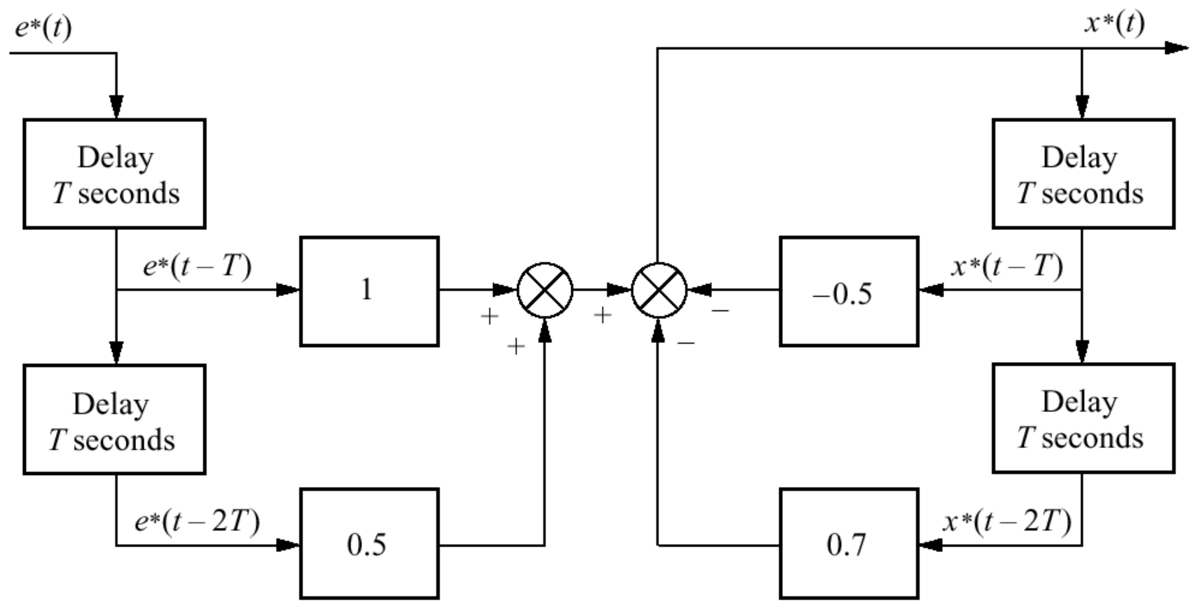
Problem: Develop a flowchart for the digital compensator defined by: $G_c(z) = \dfrac{X(z)}{E(z)} = \dfrac{z+0.5}{z^2 - 0.5z + 0.7}$

Solution:

Cross-multiply and obtain: $(z^2 - 0.5z + 0.7)X(z) = (z+0.5)E(z)$

Solve for highest power of *z* and put *X*(*z*) on the left-hand side:

$$z^2 X(z) = (z+0.5)E(z) - (-0.5z + 0.7)X(z)$$
$$\Rightarrow X(z) = (z^{-1} + 0.5z^{-2})E(z) - (-0.5z^{-1} + 0.7z^{-2})X(z)$$



Skill Assessment Ex: 13.10