Question 1.

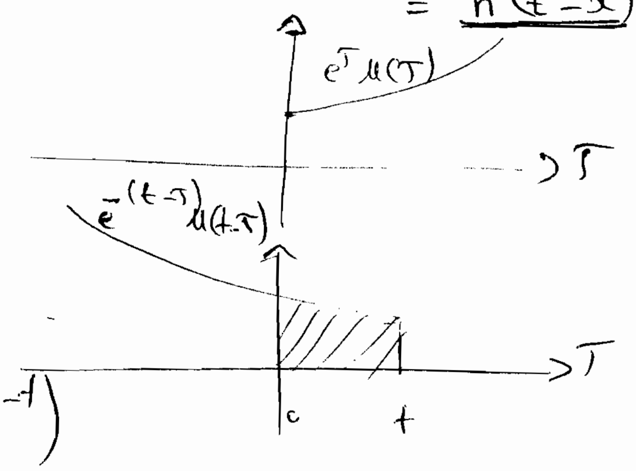
(i) $h(t) = h_1(t) * h_2(t)$

(i)

$= e^{t}\,u(t) * \left[\delta(t) - 2e^{-t}\,u(t)\right]$

(1) $= e^{t}\,u(t) * \delta(t) - 2\, e^{t}\,u(t) * e^{-t}\,u(t).$

(1) • $e^{t}\,u(t) * \delta(t) = e^{t}\,u(t)$ because $h(t) * \delta(t-x) = \underline{h(t-x)}$



• $e^{t}u(t) * e^{-t}u(t) =$

(2) $\displaystyle\int_0^t e^{\tau} e^{-(t-\tau)}\,d\tau$

$= e^{-t}\displaystyle\int_0^t e^{2\tau}\,d\tau$

$= e^{-t}\,\frac{1}{2}\,(e^{2t}-1) = \frac{1}{2}\left(e^{t}-e^{-t}\right)$

$\Rightarrow h(t) = e^{t}\,u(t) - 2\left(\frac{e^{t}-e^{-t}}{2}\right)u(t) = \underline{e^{-t}\,u(t)}$

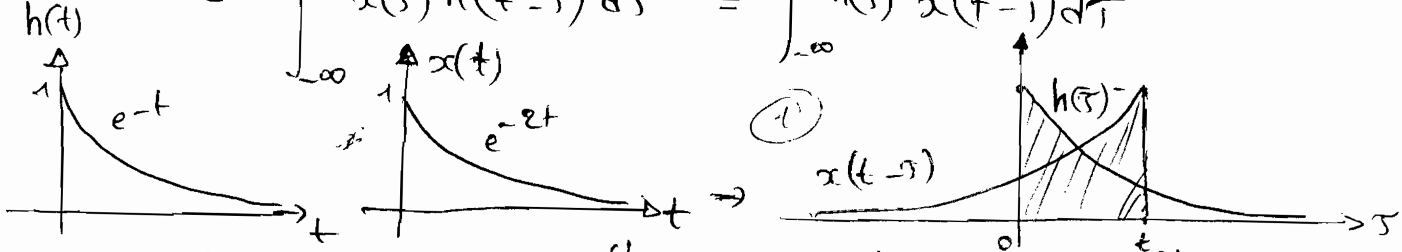
(ii) $y(t) = x(t) * h(t)$

(ii)

$= e^{-2t}\,u(t) * e^{-t}\,u(t)$

$= \displaystyle\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} h(\tau)\,x(t-\tau)\,d\tau$

(1)

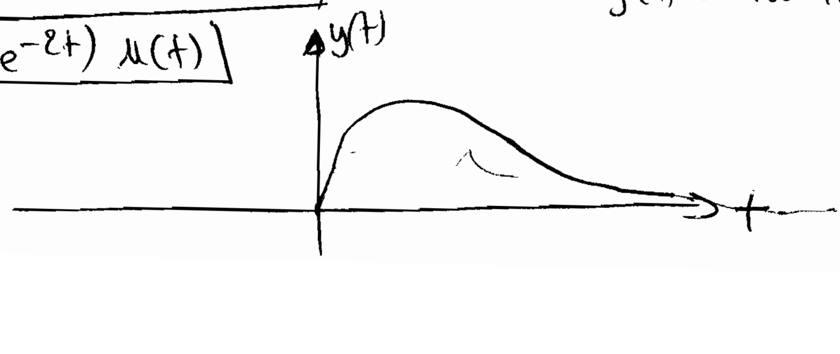


(2) $y(t) = \displaystyle\int_0^t h(\tau)\,x(t-\tau)\,d\tau = \int_0^t e^{-\tau} e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_0^t e^{\tau}\,d\tau$

$\boxed{y(t) = e^{-t} - e^{-2t} \qquad t \geq 0}$ Moreover $y(t) = 0$ for $t < 0$

$\Rightarrow \boxed{y(t) = (e^{-t} - e^{-2t})\,u(t)}$

(i)(ii)



Question 2 (a)

(b) $N_1 = 60 \qquad 3N_1 = N_2 = 180$

$N_2 = 180$

$$\omega = \frac{2\pi}{N} = \frac{2\pi}{180}$$

$$\frac{n\pi}{30} = \frac{6n\pi}{180} = 3\left(\frac{2\pi n}{18}\right)$$

$\Rightarrow N_0 = 180$

$$x[n] = \frac{e^{n\pi/30} + e^{-n\pi/30}}{2} + 2\,\frac{e^{n\pi/90} - e^{-n\pi/90}}{2j}$$

$$\frac{n\pi}{9\cancel{0}} = \left(\frac{2n\pi}{18\cancel{0}}\right)$$

$$\Rightarrow \cancel{a}[k] = \begin{cases} -1/j & k = -1 \\ 1/j & k = 1 \\ 1/2 & k = \pm 3 \\ 0 & \text{otherwise on } -8\cancel{9} \le k \le 9\cancel{0} \end{cases}$$

$$x[n] = \frac{e^{3\left(\frac{2\pi n}{180}\right)} + e^{-3\left(\frac{2n\pi}{180}\right)}}{2} + \frac{1}{j}\,\frac{e^{\frac{2n\pi}{18\cancel{0}}} - e^{-\frac{2\pi n}{18\cancel{0}}}}{}$$

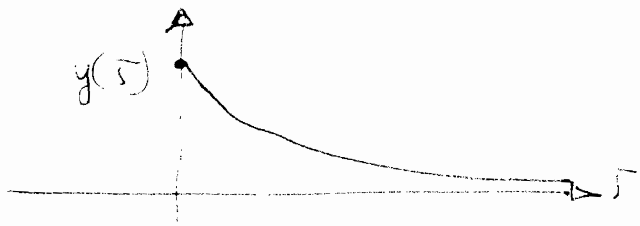
Question 2 (b)

(b) $x(t) = e^{-t}\, u(t)$

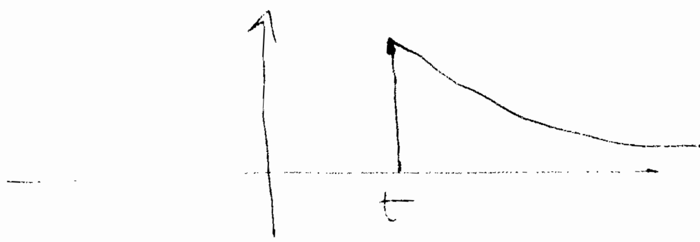
$y(t) = e^{-2t}\, u(t)$

$$R_{xy}(t) = \int_{-\infty}^{+\infty} x(\tau - t)\, y(\tau)\, d\tau$$

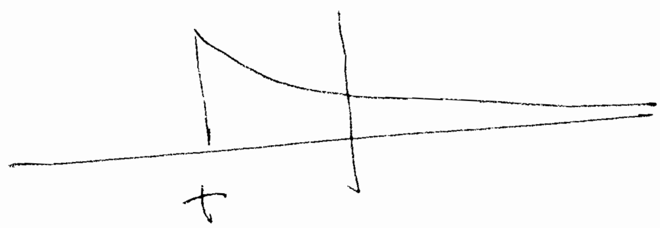
$$\begin{cases} x(\tau - t) = e^{-(\tau - t)}\, u(\tau - t) \\ y(\tau) = e^{-2\tau}\, u(\tau) \end{cases}$$



$t > 0$

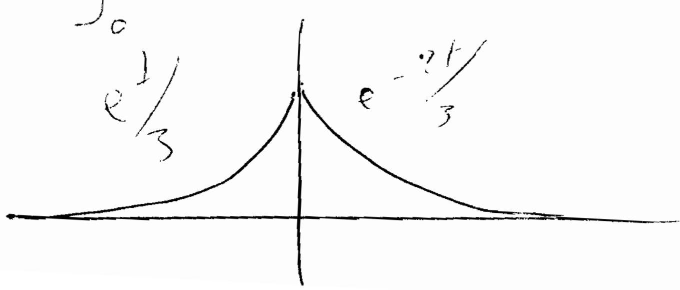


$t < 0$



$$t > 0 \Rightarrow R_{xy}(t) = \int_{t}^{\infty} e^{-(\tau - t)}\, e^{-2\tau}\, d\tau = e^{t} \cdot \left[-\frac{e^{-3\tau}}{3} \right]_{t}^{\infty} = \frac{e^{t}\, e^{-3t}}{3} = \boxed{\frac{e^{-2t}}{3}}$$

$$t < 0 \Rightarrow R_{xy}(t) = \int_{0}^{+\infty} e^{-(\tau - t)}\, e^{-2\tau}\, d\tau = e^{t}\, \frac{-1}{3}\left[e^{-3\tau} \right]_{0}^{+\infty} = \;=\; \boxed{\frac{e^{t}}{3}}$$



Problem 3 (10 points)

Consider the following periodic signal: $x[n]=\left(\frac{1}{2}\right)^n \quad 0 \le n < 10,\ x[n+10]=x[n]$

(a) Find the Fourier series coefficients a_k of x[n]

$N=10$

$$a_k = \frac{1}{10}\sum_{n=0}^{9}\left(\tfrac{1}{2}\right)^n e^{-jk\omega_0 n} = \frac{1}{10}\sum_{n=0}^{9}\left(\tfrac{1}{2}e^{-jk\omega_0}\right)^n = \frac{1}{10}\,\frac{1-\left(\frac{1}{2}e^{-jk\omega_0}\right)^{10}}{1-\frac{1}{2}e^{-jk\omega_0}} \qquad \omega_0=\frac{2\pi}{10}$$

$$= \frac{1}{10}\,\frac{1-\frac{1}{1024}}{1-\frac{1}{2}e^{-j2\pi k/10}} = \frac{1023}{10240}\,\frac{1}{1-\frac{1}{2}e^{-j2\pi k/10}}$$

(b) Find the dc component and 1st harmonic component of x[n]

$$a_0 = 2\,\frac{1023}{10240} = \frac{1023}{5120}$$

$$a_1 = \frac{1023}{10240}\,\frac{1}{1-\frac{1}{2}e^{-j2\pi/10}}$$

(c) Write down the Fourier series of x[n].

$$x[n] = \sum_{k=0}^{9}\frac{1023}{10240}\,\frac{1}{1-\frac{1}{2}e^{-j2\pi k/10}}\,e^{jkn\cdot 2\pi/10}$$

(d) Find b_k, the Fourier series coefficients of $y[n]=\begin{cases} x[n/2] & \text{if n is even} \\ 0 & \text{if n is odd}\end{cases}$

$$b_k = \frac{a_k}{2} =$$

(a) Let $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$. Find Fourier transform of $y(t) = x(-2t+7)$ (**show all steps**)

$$= x\left(-2\left(t - \tfrac{7}{2}\right)\right)$$

let $z(t) = x(-2t) \longrightarrow Z(\omega) = \frac{1}{2} X\left(-\frac{\omega}{2}\right)$

$y(t) = z\left(t - \frac{7}{2}\right) \longrightarrow Y(\omega) = Z(\omega)\, e^{-j\frac{7}{2}\omega}$

$$= \frac{1}{2} X\left(\frac{-\omega}{2}\right) e^{-j\frac{7}{2}\omega}$$

(b) The impulse response of an LTI system is given by $h(t) = e^{-3t}u(t)$. Find the output of this system if the input is $x(t) = 2e^{-5t}u(t)$. You **must** do all your analysis in the **Frequency domain (using Fourier transform).**

$X(\omega) = \dfrac{2}{5 + j\omega}$ $\qquad H(\omega) = \dfrac{1}{3 + j\omega}$

$$Y(\omega) = X(\omega) H(\omega) = \frac{2}{(5 + j\omega)(3 + j\omega)}$$

$$= \frac{-1}{5 + j\omega} + \frac{1}{3 + j\omega}$$

$$y(t) = -e^{-5t}u(t) + e^{-3t}u(t)$$