Discrete-time systems:

Consider a linear time-invariant discrete system:

$$y(k) = x(k) + 0.2\, x(k-1) + 0.5\, y(k-1)$$

Using the backward shift operator i.e $x(k-1) = z^{-1}\, x(k)$

$$y(k) = x(k) + 0.2\; Z^{-1}\, x(k) + 0.5\; Z^{-1}\, y(k)$$

or $$y(k) = \frac{1 + 0.2\, Z^{-1}}{1 - o.5\, Z^{-1}}\, x(k)$$

In Z-domain $Y(z) = G(z)\, X(z)$

Where $G(z)$ is the impulse response transfer frunction of the system which is called the Z-transform.

When a certain function is sampled at each T period its z-transform is defined as

$$E(z) = \sum_{k=o}^{\infty} e(kT)\, z^{-k}$$

This means that if the z-transform of a function is in a power (of z^{-1}) form, then the time value (e.g; $e(5T)$ equals the coefficient of Z^{-5} in the Z-transform series.

Ex. $E(Z) = 1 + 0.1\, z^{-1} - 0.3\, z^{-2} + 0.7\, z^{-5}$, $T = 1$

Then $e(o) = 1$, $e(1) = 0.1$, $e(2) = -0.3$, $e(3) = e(4) = 0$

$e(5) = 0.7$, $e(6 - \infty) = 0$.

* If a Laplace transfer of a system is given by $G_p(S)$ and a zero-order hold

device is used then

$$G(z) = (1 - z^{-1}) Z\left[\frac{G_p(s)}{s}\right]$$

$$The T.F of\ ZOH \qquad G_h(s) = \frac{1 - e^{-TS}}{s}$$

$$G(z) = \frac{y(z)}{R(z)}$$

Z-Transform theorems:

1. $Z[e(k)] = \sum_{k=o}^{\infty} e(k)\ z^{-k}$
2. $Z[e_1(k) + e_2(k)] = E_1(z) + E_2(z)$
3. $Z[a\ e(k)] = a\ E(z)$
4. $Z[e(k-n)] = z^{-n}\ E(z)$
5. $Z[e(k + n)] = z^n[E(z) - \sum_{k=o}^{n-1} e(k) z^{-k}]$
6. $Z[e^{ak}\ e(k)] = E(z\ e^{-e})$
7. Initial value

$$e(o) = \overset{liim}{Z \to \infty} E(z)$$

8. Final value

$$\lim_{n \to \infty} e(n) = \lim_{z \to 1} e\quad (z - 1)\ \ E(z)$$

The inverse Z-Transform Z^{-1} (.)

It is used to obtain values of function in time-domain only at sampled constants. It can be found by

1) Long division
2) Partial expand method

From table $$Z^{-1}\left[\frac{z}{z-a}\right] = \{a^k\}$$

3. Inversion – formula method

$$e(k) = \frac{1}{2\pi i}\oint_M E(z)\, z^{k-1}dz \qquad i = \sqrt{-1}$$

$$e(k) = \sum_{at\ pole\ of\ [E(Z)Z^{k-1}]} [residues\, of\ E(z)\, z^{k-1}]$$

(residues) = (z-a) E(z) $z^{k-1}\Big|_{z=a}$

ex. Find $Z^{-1}\left[\dfrac{1}{(z-0.1)(z-0.2)}\right]$

$$\mathrm{E(z)\ z^{k-1}} = \frac{\mathrm{k^{k-1}}}{\mathrm{(z-0-1)\,(z-0.2)}}$$, it has pole at z = 0

only when k = 0

$$r_1 = \frac{z\ z^k}{z(z-0.1)(z-0.2)}\Big|_{z=0} = \frac{1}{0.02}\ \text{included only for k = 0}$$

$$\mathrm{r_2} = \frac{\mathrm{(z-0.1)\ z^{k-1}}}{\mathrm{(z-01)\,(z-0.2)}}\Big|_{\mathrm{z=0.1}} = -(0.1)^{\mathrm{k-2}}$$

$$\mathrm{r_3} = \frac{\mathrm{(z-0.2)\ z^{k-1}}}{\mathrm{(z-0.1)\,(z-0.2)}}\Bigg|_{\mathrm{z=0.2}} = \frac{1}{0.1}(0.2)^{\mathrm{k-1}}$$

$$\Rightarrow \qquad \mathrm{e(k)} = \begin{cases} \dfrac{1}{0.02} - (0.1)^{\mathrm{k-2}} + \dfrac{1}{0.1}(0.2)^{\mathrm{k-1}} & \mathrm{k = 0} \\ -(0.1)^{\mathrm{k-2}} + 10(0.2)^{\mathrm{k-1}} & \mathrm{k > 0} \end{cases}$$

Stability:

When the transform of a discrete system transfer function is in a rational form, the zero of the denominator represents the poles of the system, in the z-domain, the system is stable if all poles are inside the unit circle.

To check stability of a discrete system either all poles are calculated and ensured to be inside the stability region (unit circle) or the bilinear transformation is used

$$z = \frac{1+(T/2)\,w}{1-(T/2)\,w}\,, \qquad W = \frac{2}{T}\,\frac{z-1}{z+1}$$

Now W $\cong$ S and therefore all points in the z-stable region is mapped is to the stable LHS of W-plane. In this case if the characteristic polynomial of the system is given in W-domain, the well known Routh-Hurwitz criterion can be used.

Analysis & Design of discrete systems:

Most tools developed for linear time-invariant continuous system are applicable in discrete system with some modifications. e.g.

- Frequency plots (Bode & Nyquist) are drawn using the relationt z = e^{JW}.
- Frequency domain design procedures for compensators are done using the bilinear transformation.

Discrtization of continuous systems:

Given the linear differential equation

$$x(t) = f[x(t)]$$

where the difference is defined as

$$x(t) = \frac{dx(t)}{dt} = \lim_{T \to o} e \frac{x[kt] - x[(k-1)T]}{T}$$

There are many approximations to the derivatives in the different equation. These include

1. Forward difference approximations

$$\left.\frac{d\,x(t)}{dt}\right|_{t=kT} = \frac{x(k+1) - x(k)}{T}$$

i.e G(z) = $\left. G(s)\right|_{S=(z-1)/T}$

2. Backward difference approximatations:

$$\left.\frac{d\,x(t)}{dt}\right|_{t=kT} = \frac{x(k) - x(k-1)}{T}$$

G(z) = $\left. G(s)\right|_{S} = \frac{(1 - z^{-1})}{T}$

3. Trapezoidal approximation

G(z) = $G(s)\big|_{s} = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$

A bilinear transformation

For nonlinear systems the forward difference approx. can be applied directly in linear systems. But when the backward difference or the trapezoidal approximations are used, a predictor-corrector such as the Runge-Kutla alg. is needed.

Example (1) descriptive the following linear system

$$\ddot{y}(t) - 0.1\,\dot{y}(t) + y(t) = u(t) \qquad y(0) = \dot{y}(.) = 0, \qquad T = 1$$

Δw. S^2 Y(s) – 0.1 SY(s) + Y(s) = U(s)

Take $S = \frac{z-1}{T}$

$(z^2 - 2z + 1)$ Y(k) – 0.1 (z-1) Y(k) + Y(k) = U(k)

y(k+2) -2y(k+1) + y(k) – 0.1y(k+1) + 0.1y(k) + y(k) = u(k)

y(k + 2) – 2.1 y(k + 1) + 2.1 y(k) = u(k)

or

y(k) – 2.1 y(k-1) + 2.1 y(k-2) = u (k – 2)

Example (2) Descriptive the non-linear system:

(T = 1)

$$y(t) + \varepsilon(y^2(t) - 1)\,\dot{y}(t) + y(t) = u(t)$$

Ans. Using the forward difference approximation then shifting in time index: (verify)

$$y(k) = u(k-2) + (2+\varepsilon)\,y(k-1) - (2+\varepsilon)\,y(k-2) - \varepsilon y^2\,(k-2)\,y(k-1) + \varepsilon\,y^3(k-2)$$