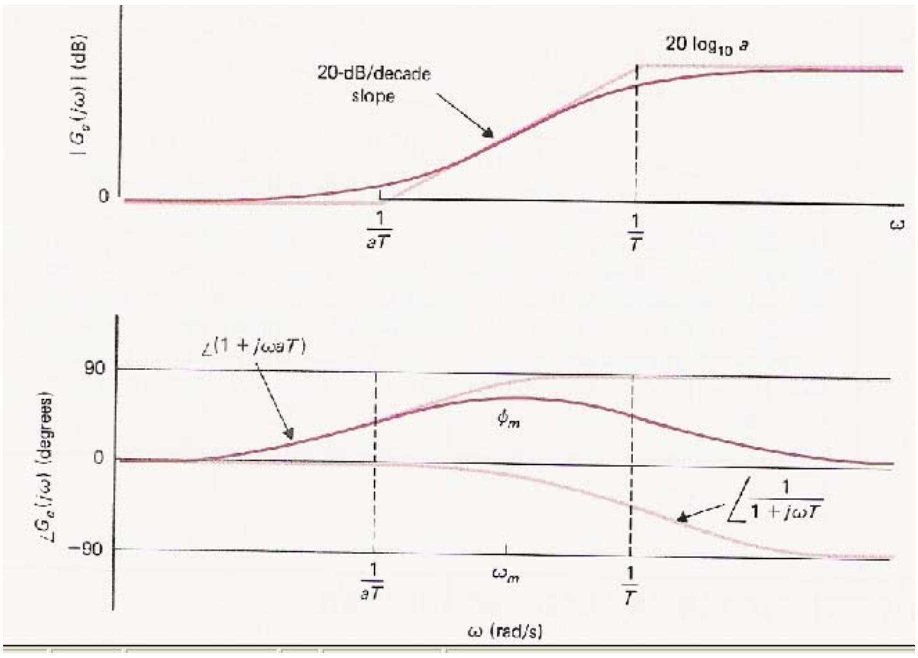
Phase-Lead Controller:

$$G_C(S) = \frac{1 + aTS}{1 + TS} \qquad a > 1$$



Figure

The objective is to increase the phase at cross over point. From the phase plot the frequency $\omega_{\max}$ at which the maximum added phase $\varphi_{\max}$ occurs is located in the middle between $\omega = \frac{1}{aT}$ & $\omega = \frac{1}{T}$ in the log scale. Hence:

$$Log\,\omega_{\max} = \frac{1}{2}\left(Log\left(\frac{1}{aT}\right) + Log\left(\frac{1}{T}\right)\right)$$

$$Log\,\omega_{\max} = \frac{1}{2}Log\left(\frac{1}{aT^2}\right) = Log\left[\left(\frac{1}{aT^2}\right)^{1/2}\right]$$

$$\Rightarrow \qquad \omega_{\max} = \frac{1}{\sqrt{aT}}$$

And $\varphi_{\max} = \tan^{-1}(a\,w_{\max}T) - \tan^{-1}(w_{\max}T)$

Let $A = \tan^{-1}(a\, w_{max} T)$, $B = \tan^{-1}(w_{max} T)$

Then

$$\text{Tan } \varphi_{max} = \tan(A - B)$$

But $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\Rightarrow \qquad \tan \varphi_{max} = \frac{a\, w_{max} T - w_{max} T}{1 + a\, w_{max}^2 T^2}$$

Substituting $w_{max} T = \dfrac{1}{\sqrt{a}}$ gives

$$\tan \varphi_{max} = \frac{a/\sqrt{a} - 1/\sqrt{a}}{1 + a/a} = \frac{a - 1}{2\sqrt{a}}$$

so when φ_{max} is known, let $\tan \varphi_{max} = f$

then

$$2\sqrt{a}\; f = a - 1 \Rightarrow 4a\; f^2 = a^2 - 2a + 1$$

$$\Rightarrow a^2 - (2 + 4f^2)\, a + 1 = 0$$

$$a = \frac{2 + 4f^2 \pm \sqrt{(2 + 4F^2)^2 - 4}}{2}$$

& discard the value less than 1

Another solution use $a = \dfrac{1 + \sin \varphi_{max}}{1 - \sin\; \varphi_{max}}$

$$\frac{Sin^2\theta}{Sin^2\theta} + \frac{Cos^2\theta}{\delta Sin^2\theta}$$

$1 + \cot^2\theta = \text{cosec}\theta$

$4a \tan^2\varphi = a^2 - 2a + 1$

$$\tan^2\varphi = \frac{a^2 - 2a + 1}{4a}$$

$$1 + \cot^2\theta = \mathrm{cosec}^2\theta$$

$$1 + \frac{4a}{a^2 - 2a + 1} = \frac{1}{\sin^2\theta} = \frac{a^2 - 2a + 1 + 49}{a^2 - 2a + 1}$$

$$\sin^2\theta = \frac{a^2 - 2a + 1}{a^2 + 2a + 1} = \frac{(a-1)^2}{(a+1)^2}$$

$$\Rightarrow \quad \sin\varphi_{max} = \frac{a-1}{a+1}$$

$$a = \frac{1 + \sin\varphi}{1 - \sin\varphi}$$

Design Procedure for phase-lead:

1) Set the open loop gain according to e_{ss} desired.
2) Plot Bode plot (gain & phase) and get gain and phase margins.
3) Set φ_{max} to the phase lead required to obtain the desired phase margin of the compensated system.

4) Determine a from φ_{max}.

5) Find the center frequency ω_{max} (it is the frequency at which the magnitude of the uncompensated system equals ($-10\log_{10} a$) to cancel ($+10\log a$) added by the compensator to locate the gain-crossover frequency.

6) Calculate T from a & ω_{max}.

7) Check the perform of the compensated system (φ_m & G_m) change φ_{max} if necessary.

Notes on Phase-Lead:

1. The phase margin is improved (and gain margin).
2. The bandwidth is increased.
3. Less overshoot and reduced rise time.

Phase lag controllers:

$$G_c(S) = \frac{1 + aTS}{1 + TS} \qquad a < 1$$

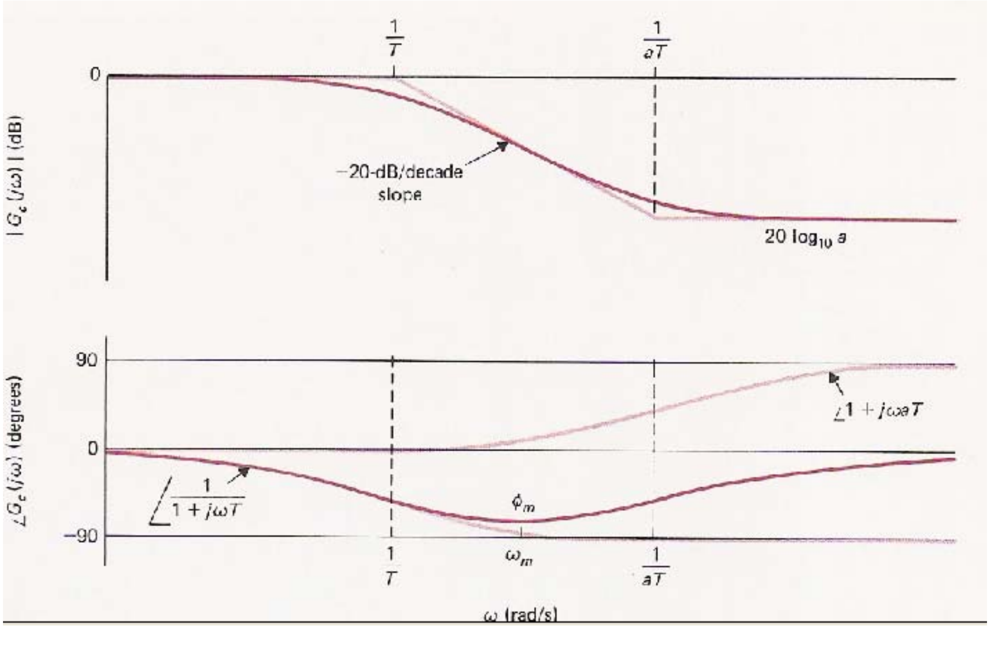


Figure here

As before

$$\text{Sin } \varphi_{\max} = \frac{a-1}{a+1}$$

The objective here is to move the gain-crossover frequency to a lower frequency while keeping the phase curve of the bode plot relatively unchanged at this frequency.

Design Procedure:

1) Set the open loop gain of the uncompensated system according to the e_{ss} requirement.
2) Plot the Bode plot and determine the phase & gain margins.
3) According to a specified phase margin to be achieved, the frequency corresponds to this phase margin is located which will be the new gain-cross over frequency ω_c'.
4) To bring down the magnitude curve down to 0-dB at ω_c, a gain is provided by the compensator (attenuation) i.e.

$$\left|G_p(J\omega_c)\right| = -20\log(a)\ dB \qquad a < 1 \qquad \text{or}$$

$$a = 10^{-16(j\omega_c)/20} \qquad a < 1$$

5) In order that the change in phase due to the compensate at ω_c is kept low, the upper corner frequency $\frac{1}{\text{aT}}$ is placed far below the cross-over frequency ω_c (note $\frac{1}{\text{aT}}$ should not be too small to avoid large decrease in bandwidth). Usually

$$\frac{1}{aT} = \frac{\omega_c}{10} \; rad/s \qquad \text{hence} \qquad T = \frac{10}{\omega_c a}$$

6) Check G_m & Q_m

Notes on lag compensators:

- Improved relative stability due to gain attenuation near and above cross-over frequency.
- The bandwidth is decreased.
- Usually slower rise & setting times.

NON-LINEAR SYSTEMS:

Introduction:

Linear systems are systems that obey the superposition theorem. Assume the system is completely described by the input relationship,

c(t) = f(r). This theorem states that the output response of a linear time-invariant system to a number of simultaneously applied inputs is equal to the summation of the system responses when each input is applied individually, i.e.

$$f[a_1 r_1 + a_2 r_2] = a_1 f[r_1] + a_2 f[r_2]$$

where a_1 and a_2 are: constants

for example: (static system)

(I) y(t) = a x(t)

$f(x) = a\,x(t)$

$f(a_1r_1) = a_1ar_1,\quad f(a_2r_2) = a_2a\,r_2,\quad f(a_1r_1 + a_2r_2) = a(a_1r_1 + a_2r_2)$

$\Rightarrow \quad f(r_1 + r_2) = f(r_1) + f(r_2)$ linear system

(II) is $y(t) = a\,x(t) + b$ a linear system?

$f(a_1r_1) = a\,(a_1r_1) + b$

$f(a_2r_2) = a\,(a_2\,r_2) + b$

$f(a_1r_1 + a_2\,r_2) = a\,(a_1\,r_1 + a_2\,r_2) + b$

$f(a_1\,r_1) + f(a_2\,r_2) = a(a_1\,r_1 + a_1\,r_2) + 2\,b + f(a_1\,r_1 + a_2r_2)$

$\Rightarrow$ system is not a linear system

The examples are:

$F(x) = x^2(x),\qquad f(x) = \sin(x)\qquad e^x$……..

Dynamic system:

Let $\dot{x}(t) = A\,x(t) + B\,u(t)$

$y = C\,x(t) + D\,u(t)$

The solution is $y(s) = \overbrace{[C(SI\text{-}A)^{-1}\,B + D]}^{G(s)}\,u(s)$

$\Rightarrow \quad f(a_1\,R_1) = \int^{-1}\ [G(s)\,a_1\,R_1(s)]$

$f(a_2\,R_2) = \int^{-1}\ [G(s)\,a_2\,R_2(s)]$

$f(a_1R_1 + a_2\,R_2) = \int^{-1}\ [G(s)\,(a_1\,R_1 + a_2\,R_2)]$

$= \int^{1} [G(s)\,a_1R_1] + \int^{-1}\ [a_2R_2\ G(s)]$

$= f(a_1R_1) + f(a_2\,R_2)$

A sub system of the above system is

$$\dot{y}(t) + 2y(t) = \dot{u}(t) + 3u(t)$$

But for the system

$$\dot{y}(t) + 2y(t) = u^2(t)$$

or

$$\dot{y}(t) + 2x(t)\, y(t) = u(t)$$

$$\dot{y}(t) + 2y^2(t) = x(t)\, y(t) + x(t)$$

They all do not satisfy the superposition theorem and therefore they are not linear systems.

Now linear systems are, simply, systems that do not obay the superposition theorem.

Some types of non-linear systems:

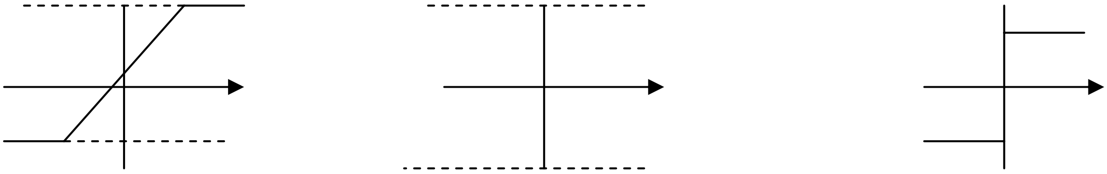
Non-linear systems are generally given by

$$y(t) = f(t, \theta, x)$$

where f is a nonlinear function of the time t, parameter θ and the input x.

* Memoryless (Static) systems:

The most common nonlinear effect is saturation



Figure

* Dynamic Non-linear systems:

i) Wiener system:

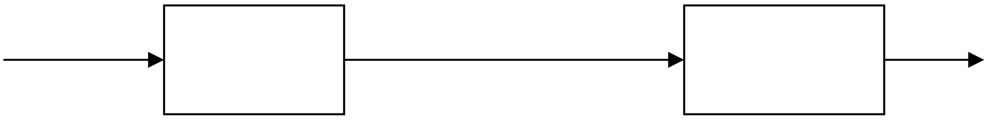


Figure here

Ex. $\dot{w} + 2w = \dot{x}(t) + 3x(t)$

$y(t) = 0.2w^2(t) + 0.5\ w(t)$

ii) Hammerstein system:

Figure

Ex.

$w = x(t) + 0.5\ x^2(t)$

$\ddot{y}(t) + 2\dot{y}(t) + 3y(t) = \dot{w}(t) + 0.5\ \ w(t)$

iii General separable non-linear system

Figure

iv) Non separable dynamic nonlinear systems:

Ex. $\ddot{y}^2(t) + 2\dot{y}(t)\,\dot{x}(t) + \sqrt{x(t)\,y(t)} = x^{1/3}(t)$

Linearization of non-linear systems:

Unforced linear systems with $|A| \neq 0$ have only one equilibrium point x_o ($x_o = 0$). But non-linear systems my have more than one equilibrium points. For example:

$$\ddot{\theta}_o + a\,\dot{\theta}_o + K\sin\theta_o = 0$$

let $x_1 = \theta_o$ and $x_2 = \dot{x}_1 = \dot{\theta}_o$ then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -k\sin x_1 - a\,x_2$$

The equilibrium points exist at $\dot{x} = f(x_o) = 0$

or $x_2 = 0$ and $x_1 = k\pi$ (k is an integer)

(multiple points)

Non-linear system are linearized only if the changes of the state around the operating points are very small. When the changes are not small enough the linear approximation of the nonlinear system is not valid. Linearization is usually carried out for a specified equilibrium point, if it changes the non-linear system has to be linearized again.

In general a forced non-linear system is given by:

$$\dot{x} = f(x,u)$$

Where x is the state vector and y is the input. Then in a small neighborhood about each of the equilibrium points the non-linear system acts as a linear system. Therefore for equilibrium points x_0 let $x = x_0 + x^*$ where x^* represents state deviation from x_o, and assuming that x^* is restricted to a very small region around. x_0. Using the Taylor series while higher order terms are neglected:

$$\dot{x}^* = \begin{bmatrix} \frac{\partial f_1(x_1 u)}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_{n1}}{\partial x_n} \end{bmatrix}_{\substack{x=x_o \\ u=u_o}} x^* \quad + \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial u_1} & \frac{\partial f_1(x,u)}{\partial u_r} \\ \frac{\partial f_n(x,u)}{\partial u_,} & \frac{\partial f_n(x,u)}{\partial u_r} \end{bmatrix}_{\substack{x=x_o \\ u=u_o}} u^*$$

$$= J_x\, x^* + J_u\, u^*$$

where J is called the Jacobian matrix and is evaluated at (x_o,u_o).

Ex. $\ddot{\theta} + a\dot{\theta} + k\sin\theta = 0$

$$\dot{x}_1 = x_2 \qquad = f_1(x)$$

$$\dot{x}_2 = -k\sin x_1 - ax_2 = f_2(x)$$

$$x = x_0 + x^* \text{perturbation}$$

$$f(x_o) = 0 \quad \Rightarrow x_2 = 0 \quad \& \, x_1 = k\pi$$

$$\dot{x}^* = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} x^*$$

For equilibrium points $x_1 = x_2 = 0$

$$\dot{x}^* = \begin{bmatrix} 0 & 1 \\ -k\cos x & -a \end{bmatrix}_{x=x_o} x^* = \begin{bmatrix} 0 & 1 \\ -k & -a \end{bmatrix} x^*$$

This system is always stable since the eigenvalues are :

$$\lambda_{1,2} = \frac{-a}{2} \pm \sqrt{(a/2)^2 - k}$$

For the equilibrium point $x_2 = 0$, $x_1 = \pi$

$$\dot{x}^* = \begin{bmatrix} 0 & 1 \\ k & -a \end{bmatrix} x^*$$

This system may or may not be stable depending on the values of a & K.

$$\lambda_{1,2} = \frac{-a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 + K}$$