4. Modules Over Principal Rings

We have established that free modules over commutative rings like vector spaces have a well-defined rank. Now let F be a free module over a commutative ring R, it seem natural to ask questions like:

- Is every submodule of F also free?
- Suppose further that F has finite rank, then is every submodule free in this case? Is it finitely generated?
- If F_1 is a submodule of F so that both modules are free with well-defined ranks, is rank of F_1 less than or equal the rank of F?

Unfortunately, the answer to all these questions is in general no. But fortunately, we do not have to restrict to working over fields to some nice related theory.

Exercise 1. Give examples to show the vector space properties above do not hold for modules. That is, find a submodule M of an R-module F so that:

(a) F is free and M is not free.
(b) F is free finitely generated, but M is not finitely generated.

©Both F and M are free with rank $M >$ rank F.
In this section we assume that R is a PID (Principal Ideal Domain).

2. Let F be a free module over R, a PID. Then every submodule M of F is free with $\operatorname{rank} M \leq \operatorname{rank} F$.

For simplicity we prove the theorem for rank$F$ finite. By induction on $\operatorname{rank} F$. IF $\operatorname{rank} F = 1$, then $F \cong R$. A submodule M of F is then isomorphic to an ideal of R, a PID. Hence M is free and rank ≤ 1. Assume the statement is true for rank $F \leq n-1$. Suppose $\operatorname{rank} F = n$, say $F = \langle \{x_1, x_2, \ldots, x_n\} \rangle$. For $k = 1, 2, \ldots, n$ consider the free submodule $F_k = \langle \{x_1, x_2, \ldots, x_k\} \rangle$. That is

$$F_1 = \langle \{x_1\} \rangle, \quad F_2 = \langle \{x_1, x_2\} \rangle, \quad \ldots, \quad F_{n-1} = \langle \{x_1, x_2, \ldots, x_{n-1}\} \rangle, \quad F_n = F$$

Since $F_{n-1} \subseteq F_n$, then $M \cap F_{n-1}$ is a submodule of M, inducing an exact sequence.

$$0 \to M \cap F_{n-1} \to M \to M/M \cap F_{n-1} \to 0$$

Since $M/M \cap F_{n-1}$ is a submodule of F_n/F_{n-1} which is isomorphic to R, then $M/M \cap F_{n-1}$ is free of rank ≤ 1. This implies that the exact sequence splits, hence

$$M \cong M \cap F_{n-1} \oplus M/M \cap F_{n-1}$$

Finally, $M \cap F_{n-1}$ is a submodule of F_{n-1} free of rank $n-1$, by induction $M \cap F_{n-1}$ is free with rank $\leq n-1$. Therefore, M is a free module, and with $\operatorname{rank} M \leq n - 1 + 1 = n$. ⬟

Exercise 3. Show that the theorem is not true if R is an integral domain which is not a PID.

(Hint: Consider submodules of $R = \mathbb{C}[x,y]$ or $\mathbb{Z}[x]$.)

Corollary 4. If E is a finitely generated, say by n elements over R, a PID. Then any submodule E' of E is finitely generated by at most n.

Proof. Since E has a set of n generators, then there is an exact sequence $R^n \xrightarrow{\varphi} E \to 0$. Now $\varphi^{-1}(E')$ is a submodule of R^n a free module of rank n. By 00, $\varphi^{-1}(E')$ is free of rank $\leq n$. From the exact sequence $\varphi^{-1}(E') \to E' \to 0$, E' is generated by at most n elements. ⬟

Definition 5. Let R be a domain and M an R-module. Then

- An element $m \in M$ is called a **torsion** element if $rm = 0$ for some $r \in R^{\times}$. The set of all torsion elements of M is denoted M_{tor}.
- If $M_{\text{tor}} = M$, then M is called a **torsion module**. If $M_{\text{tor}} = 0$, then M is said to be torsion free.

Exercise 6. Let M be module over a domain R. Show that

a. M_{tor} is a submodule of M.
b. M/M_{tor} is torsion free.
c. If M is free over R, then M must torsion free.

The converse in **c**. is not true.

Proposition 7. A torsion free finitely generated module M over a PID is free.

Proof. Assume $M = \langle \{x_1, x_2 \dots, x_n\} \rangle$. Let $B = \{y_1, y_2, \dots, y_k\} \subseteq \{x_1, x_2 \dots, x_n\}$ be a maximal linearly independent subset. Then each x_i satisfy a relation $a_i x_i = \sum x_i y_i$, i.e. $a_i x_i \in \langle B \rangle$. If $N = \langle B \rangle \subseteq M$, then N is free. Let $a = \prod_{i=1}^{n} a_i$, then as $a_i x_i \in N \Rightarrow a x_i \in N \Rightarrow aM \subseteq N$. Therefore, aM is free. Since M is torsion free, then $T_a : M \to M : T_a(m) = am$ is injective, so $M \cong aM$ and M is free. ⬟

Theorem 8. Let M be finitely generated over a PID R. There exists a free submodule F of M such that M is a direct sum $M = M_{\text{tor}} \oplus F$.

Consider the exact sequence $0 \to M_{\text{tor}} \to M \to M/M_{\text{tor}} \to 0$ splits. Since M is finitely generated, then M/M_{tor} is finitely generated but M/M_{tor} is also torsion free. Hence M/M_{tor} is free. Since a free module is projective, then the sequence splits and $M \cong M_{\text{tor}} \oplus M/M_{\text{tor}}$. ⬟

The rank of the free module F is uniquely determined by M, since if F' is another free module with $M = M_{\text{tor}} \oplus F'$ then F' must also be isomorphic with M/M_{tor}.

Definition 9. The rank of a finitely generated module M over a PID is defined to be the rank of the free module F in Theorem 8.

Corollary 10. For a finitely generated module M over a PID, the torsion submodule M_{tor} is a summand of M. The following example shows that this fact requires finite generation in general.

If $R = \mathbb{Z}$ and $M = \prod_{n=1}^{\infty} \mathbb{Z}_n$, then one can show that the torsion submodule is $M_{\text{tor}} = \oplus_{n=1}^{\infty}\mathbb{Z}_n$ which clearly not a direct summand.

Corollary 11. A finitely generated module M over a PID is free *iff* it is torsion free.

Proof. Any free module over a domain is torsion free. Now if M is torsion free, then $M_{\text{tor}} = 0$ and since M is finitely generated, we have $M = M_{\text{tor}} \oplus F = 0 \oplus F = F$ is free. ⬟

Note that over a PID a torsion free module is not necessarily free. For example, as a $\mathbb{Z}$-module $\mathbb{Q}$ is torsion free but not free.

Exercise 12. Show that $\mathbb{Q}$ is torsion free which is not free.

Corollary 13. A finitely generated module M over a PID is free *iff* it is projective.

Proof. We know free $\Rightarrow$ projective in general. Let P be a finitely generated projective module over a PID R. Then P is a direct summand of a free R-module, hence must be torsion free. ⬟

In fact, Corollary 13 is true even if M is not finitely generated. Since a projective module is isomorphic to a submodule of a free R-module which by 0 2 must be free. This proves the following corollary. ⬟

Corollary 14. Over a PID a module M is projective *iff* it is free.